The Big Bang - Cosmic Expansion

\[ \frac{dR}{dt} = \sqrt{\frac{8\pi G \rho}{3R}} \frac{\Lambda}{3} R^2 \]

According to Big Bang theory, the scale of the universe increases with time at a rate that depends on the density of matter, \( \rho \), and the size of the cosmological constant, \( \Lambda \). This is defined by the fundamental equation to the left.

**Problem 1** - Determine the general form of the integral that relates the time, \( t \), to the value of the scale factor, \( R \); Solve the integral for the time, \( t \), but do not solve the integral for \( R \).

**Problem 2** - Transform the integral for \( R \) to a new variable, \( U \), such that \( U = \left( \frac{A}{C} \right)^{1/3} R \) where \( A = \Lambda / 3 \) and \( C = 8\pi G \rho / 3 \).

**Problem 3** - Solve the integral for two special cases A) The Inflationary Universe case where \( U >> 1 \) and B) the matter-dominated universe case where \( U << 1 \).

**Problem 4** - Hubble's Constant is a measure of the rate of expansion of the universe. It is defined as \( H = 1/R \left( \frac{dR}{dt} \right) \). Find the formula for Hubble's Constant for the two cosmological cases described in Problem 3.
Problem 1 - The integral equation is then

\[ \int \frac{1}{t} \, dt = \int \frac{dR}{\sqrt{\frac{8\pi G \rho}{3R} + \frac{\Lambda}{3} R^2}} \]

Problem 2  First clean up the rather cumbersome radical expression so that it only involves \( R \) to positive powers and the constants \( A \) and \( C \), by factoring out \( (1/R)^{1/2} \) to get \( (1/R)^{1/2} (C + A R^3)^{1/2} \). Factor out the constant \( C \) from the square-root so that the denominator of the integrand becomes \( (1/R)^{1/2} C^{1/2} (1 + A/C R^3)^{1/2} \) and replace with \( U = (A/C)^{1/3} R \) to get

\[ C^{1/2} (A/C)^{1/6} U^{-1/2} (1 + U^3)^{1/2} \]

Note that we have also transformed the \( (1/R)^{1/2} \) factor by replacing it with \( (A/C)^{1/6} U^{-1/2} \). Since \( dU = (A/C)^{1/3} dR \), we can now re-write the complete integrand as

\[ (1/C)^{1/2} (C/A)^{1/6} (C/A)^{1/3} U^{1/2} dU / (1 + U^3)^{1/2} \]

After combining the constants \( A \) and \( C \) and replacing them with their definitions the integrand simplifies to \( (3/\Lambda)^{1/2} U^{1/2} dU / (1 + U^3)^{1/2} \) and the integral becomes

\[ t = \sqrt{\frac{3}{\Lambda}} \int \frac{U^{1/2} dU}{\sqrt{U^3 + 1}} \]

Problem 3 A) If \( U > 1 \), then the term under the square-root is essentially \( U^3 \), so we get \( U^{1/2} / U^{3/2} = 1/U \). This leads to an integrand of \( (3/\Lambda)^{1/2} 1/U \, dU \) which is a fundamental integral whose solution is \( t = (3/\Lambda)^{1/2} \ln U + C \). This can be re-written as \( U(t) = e^{[(\Lambda/3)^{1/2} t]} \). From the definition for \( U \) we get

\[ R(t) = \left( \frac{8\pi G \rho}{\Lambda} \right)^{1/3} e^{\left( \frac{\Lambda}{3} \right)^{1/2} t} \]

This represents a universe that expands at an exponential rate because of the positive pressure provided by the cosmological constant - a property of the energy of empty space. This solution is thought to describe our universe during its' inflationary' era shortly after the Big Bang.

Problem 3 B) In this case, \( U << 1 \) so the term under the square-root is essentially 1, and the integrand becomes \( (3/\Lambda)^{1/2} U^{1/2} \, dU \). This is easily integrated to get \( t = (3/\Lambda)^{1/2} U^{3/2} \). After substituting for the definition of \( U \) we get \( t = (3/\Lambda)^{1/2} (\Lambda/8\pi G \rho)^{1/2} R^{3/2} \) so that

\[ t = (3/8\pi G \rho)^{1/2} R^{3/2} \]

This can be easily inverted to get

\[ R(t) = \left( \frac{8\pi G \rho}{\Lambda} \right)^{1/3} t^{2/3} \]

This solution is the 'matter-dominated' cosmology represented by Big Bang cosmology, and applies to the modern expansion of the universe.

Problem 4 A) \( H = (\Lambda/3)^{1/2} \) and B) \( H = 2/3 \, (1/t) \). In the inflationary case, the rate of expansion is constant in time, but in the matter-dominated case, the expansion rate decreases in proportion to the age of the universe.