Ion rocket motors provide a small but steady thrust, which causes a spacecraft to accelerate. The shape of the orbit for the spacecraft as it undergoes constant acceleration is a spiral path. The length of this path can be computed using calculus.

The arc length integral can be written in polar coordinates where the function, \( y = F(x) \) is replaced by the polar function \( r(\theta) \).

Because the integrand is generally a messy one for most realistic cases, in the following problems, we will explore some simpler approximations.

The Dawn spacecraft was launched on September 27, 2007, and will take a spiral journey to visit the asteroid Vesta in February 2015. Earth is located at a distance of 1.0 Astronomical Units from the Sun (1 AU = 150 million kilometers) and Vesta is located 2.36 AU from the Sun. The journey will take about 66,000 hours and make about 3 loops around Earth's orbit in its outward spiral as shown in the figure to the left.

**Problem 1** - Suppose that the Dawn spacecraft travels at a constant outward speed from Earth's orbit. If we approximate the motion of the spacecraft by \( X = R \cos \theta \), \( Y = R \sin \theta \) and \( R = 1 + 0.08 \theta \), where the angular measure is in radians, show that the path taken by Dawn is a simple spiral.

**Problem 2** - From the equation for \( R(\theta) \), compute the total path length of the spiral from \( R = 1.0 \) to \( R = 2.36 \) AU, and give the answer in kilometers. About what is the spacecraft's average speed during the journey in kilometers/hour? [Note: Feel free to use a Table of Integrals!]

**Problem 3** - The previous two problems were purely 'kinematic' which means that the spiral path was determined, not by the action of physical forces, but by employing a mathematical approximation. The equation for \( R(\theta) \) is based on constant-speed motion, and not upon actual accelerations caused by gravity or the action of ion engine itself. Let's improve this kinematic model by approximating the radial motion by a uniform acceleration given by \( R(\theta) = 1/2 \ A \ \theta^2 \) where we will approximate the net acceleration of the spacecraft in its journey as \( A = 0.009 \). What is the total distance traveled by Dawn in kilometers, and its average speed in kilometers/hour?
Problem 1) Answer computed using Excel spreadsheet.

Problem 2: \( R = 1.0 + 0.08 \theta \) and so \( dR/d\theta = 0.08 \) and \( d\theta/dR = 12.5 \). The integrand becomes \((1 + 156R^2)^{1/2}\) \(dR\).

If we use the substitution \( U = 12.5R \) \(dU = 12.5 \) \(dR\) and the integrand becomes 0.08 \((1 + U^2)^{1/2}\) \(dU\). A table of integrals yields the answer

\[
\frac{1}{2} \left[ U \left(1 + U^2\right)^{1/2} + \ln \left(U + \left(1 + U^2\right)^{1/2}\right) \right] = 1/2
\]

Astronomical Units or 28.6 x 150 million km = 4.3 billion kilometers! The averages speed would be about 4.3 billion/66000 hrs = 65,100 kilometers/hour.

Problem 3 - \( dR/d\theta = A \theta \) so that \( d\theta/dR = 1/(A \theta) \).

From \( R(\theta) \), we can re-write \( d\theta/dR \) solely in terms of \( R \) as \( d\theta/dR = \left(1/(2Ar)\right)^{1/2} \) so that the integrand becomes \((1 + R/(2A))^{1/2}\) \(dR\).

Unlike the integral in Problem 1, this integral can be easily performed by noting that if we substitute

\[
U = 1 + R/(2A), \text{ and } dU = dR/2A,
\]

we get the integrand \(2A U^{1/2} \) \(dU\) and so

\[
S = \left(\frac{4A}{3}\right) U^{3/2} + C.
\]

The limits to this integral are \(U_i = 1 + 1.0/2A = 56.\) and \(U_f = 1 + 2.36/2A = 132.\)

Then the definite integral becomes

\[
S = \left(\frac{4 \times 0.009}{3}\right) \left[132^{3/2} - 56^{3/2}\right] = 0.012 \left[1516 - 419\right] = 13.2 \text{ AU}.
\]

Since 1 AU = 150 million km, the spiral path has a length of 2.0 billion kilometers. The averages speed would be about 2.0 billion km/66000 hours = 30,300 km/hour. The trip takes less time because the 'kinematic' motion is speeded up towards the end of the journey.