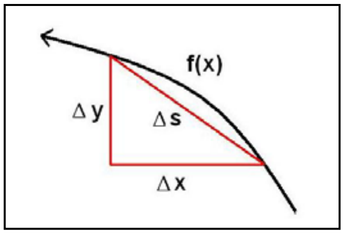
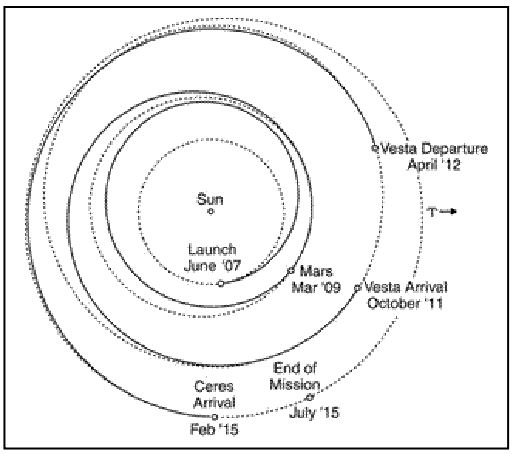



$$s = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Ion rocket motors provide a small but steady thrust, which causes a spacecraft to accelerate. The shape of the orbit for the spacecraft as it undergoes constant acceleration is a spiral path. The length of this path can be computed using calculus.

The arc length integral can be written in polar coordinates where the function, y = F(x) is replaced by the polar function **r(θ).**

Because the integrand is generally a messy one for most realistic cases, in the following problems, we will explore some simpler approximations.




The Dawn spacecraft was launched on September 27, 2007, and will take a spiral journey to visit the asteroid Vesta in February 2015. Earth is located at a distance of 1.0 Astronomical Units from the Sun (1 AU = 150 million kilometers) and Vesta is located 2.36 AU from the Sun. The journey will take about 66,000 hours and make about 3 loops around Earth's orbit in its outward spiral as shown in the figure to the left.

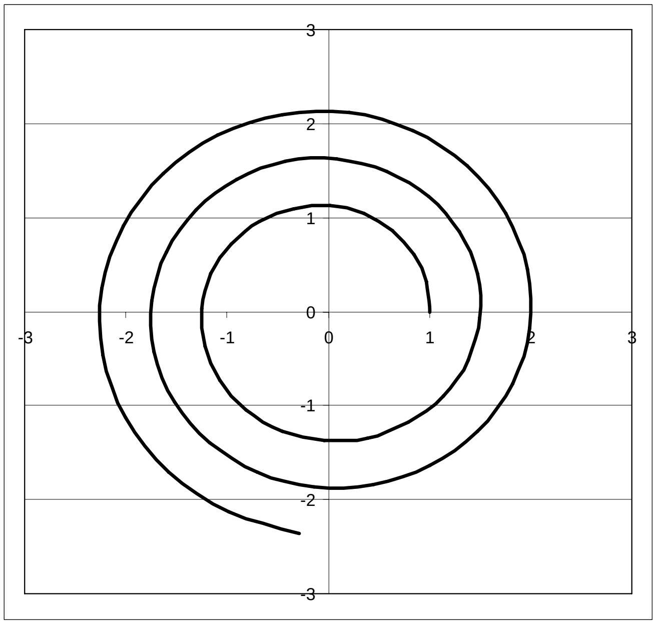
**Problem 1** - Suppose that the Dawn spacecraft travels at a constant outward speed from Earth's orbit. If we approximate the motion of the spacecraft by $X = R\cos\theta$, $Y = R\sin\theta$ and $R = 1 + 0.08\,\theta$, where the angular measure is in radians, show that the path taken by Dawn is a simple spiral.

**Problem 2** - From the equation for R(θ), compute the total path length of the spiral from R=1.0 to R = 2.36 AU, and give the answer in kilometers. About what is the spacecraft's average speed during the journey in kilometers/hour? [Note: Feel free to use a Table of Integrals!]

**Problem 3** - The previous two problems were purely 'kinematic' which means that the spiral path was determined, not by the action of physical forces, but by employing a mathematical approximation. The equation for R(θ) is based on constant-speed motion, and not upon actual accelerations caused by gravity or the action of ion engine itself. Let's improve this kinematic model by approximating the radial motion by a uniform acceleration given by $R(\theta) = 1/2\, A\, \theta^2$ where we will approximate the net acceleration of the spacecraft in its journey as A = 0.009. What is the total distance traveled by Dawn in kilometers, and its average speed in kilometers/hour?

# Answer Key

**Problem 1) Answer computed using Excel spreadsheet.**

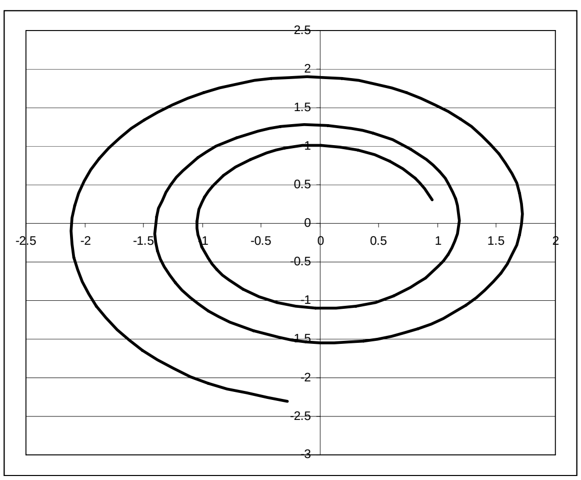


**Problem 2:** R = 1.0 + 0.08 θ and so dR/dθ = 0.08 and dθ/dR = 12.5. The integrand becomes $( 1+ 156R^2 )^{1/2}$ dR .

If we use the substitution U = 12.5R dU = 12.5 dR and the integrand becomes 0.08 $(1 + U^2)^{1/2}$ dU. A table of integrals yields the answer

$1/2 [ U (1 + U^2)^{1/2} + \ln (U + (1+ U^2)^{1/2} ).$

The limits to the integral are Ui = 12.5 x 1.0 = 12.5 and Uf = 12.5 x 2.36 = 29.5, and when the integral is evaluated we get 1/25 [ 29.5 (29.5) + ln (29.5 + (29.5) ) - 12.5 (12.5) - ln(12.5 + (12.5))] = 1/25 (870 + 4.1 - 156 - 3.2) = 28.6 Astronomical Units or 28.6 x 150 million km = 4.3 billion kilometers! The averages speed would be about 4.3 billion/66000 hrs = **65,100 kilometers/hour.**



**Problem 3 -** dR/dθ = A θ so that
dθ/dR = 1/(A θ).

From R(θ), we can re-write dθ/dR solely in terms of R as dθ/dR = $(1/(2Ar))^{1/2}$ so that the integrand becomes $(1 + R/(2A))^{1/2}$ dR.

Unlike the integral in Problem 1 ,this integral can be easily performed by noting that if we substitute

U = 1 + R/(2A), and dU = dR/2A,

we get the integrand 2A $U^{1/2}$ dU and so
S = (4A/3) $U^{3/2}$ + C.

The limits to this integral are Ui = 1 + 1.0/2A = 56. and Uf = 1 + 2.36/2A = 132.

Then the definite integral becomes
S = (4 x 0.009/3) [ $132^{3/2}$ - $56^{3/2}$] = 0.012 [ 1516 - 419] = 13.2 AU .Since 1 AU = 150 million km, the spiral path has a length of **2.0 billion kilometers.** The averages speed would be about 2.0 billion km/66000 hours = **30,300 km/hour**. The trip takes less time because the 'kinematic' motion is speeded up towards the end of the journey.