Rotation Velocity of a Galaxy

Spiral galaxy M-101 showing its bright nucleus and spiral arms. The radius of M-101 is about 90,000 light years, which corresponds to x=9 in the formula for V(x). (Hubble image)

Stars orbit the center of a galaxy with speeds that decrease as their orbital distances increase. A simple function, V(x) can model the orbital speeds of stars as a function of their distance, x, from the nucleus of the galaxy:

$$V(x) = \frac{350x}{\left(1+x^2\right)^{\frac{3}{4}}}$$

For example: At a distance of 10,000 light years from the center, x = 1.0 and the rotation speed is V(1.0) = 208 kilometers/sec.

Problem 1 – For small x (i.e. x < 1), what is the limiting form of V(x)?

Problem 2 – For large x, (i.e. x > 1) what is the limiting form of V(x)?

Problem 3 - The radius of M-101 is 90,000 light years. How fast are stars orbiting the center of M-101 according to V(x)? (Hint: At a radius of 90,000 light years, x=9.0. If the units of V(x) are kilometers/sec, what is V(x) at x = 9.0?)

Problem 4 – For what value of x is V(x) maximum?

Problem 5 – For x=1 the physical distance is 10,000 light years. How many years does it take a star to complete one circular orbit at x=1.0 if 1 light year equals 9.5×10^{12} km, and there are 3.1×10^{7} seconds in a year?

Note: This example of V(x) is for galaxies in which most of the mass is concentrated within their central regions (x < 1), however, astronomers know that this model is not completely accurate. Beyond x = 1, the rotation speeds for some galaxies, including the Milky Way, do not decrease rapidly as suggested by V(x), but actually remain constant. This implies that some galaxies contain substantial amounts of 'Dark Matter' that is not in the form of stars or other known forms of matter.

Problem 1 – For small x, what is the limiting form of V(x)?
Answer: The denominator approaches 1 and so **V(x) = 350x**

Problem 2 – For large x, what is the limiting form of V(x)?
Answer: In the denominator, x^2 dominates over 1 so the denominator approaches $x^{3/2}$ and so $V(x) = 350x/x^{3/2}$ becomes:

$$V(x) = \frac{350}{\sqrt{x}}$$

Problem 3 - The radius of M-101 is 90,000 light years. How fast are stars orbiting the center of M-101 according to V(x)? (Hint: At a radius of 90,000 light years, x=9.0. If the units of V(x) are kilometers/sec, what is V(x) at x = 9.0?)
Answer:

$$V(9) = \frac{350(9)}{(1+9^2)^{\frac{3}{4}}}$$ = **26 kilometers/sec**

Note: X is a pure number. It represents the ratio X = (d /10,000 light years) where d is a physical distance in units of light years. Example: at a physical distance of 40,000 light years from the center of the galaxy, x = 40,000 LY/10,000 LY so x = 4.0. The rotation speed of stars at this distance is just $V(4) = 350(4)/(1+4^2)^{\mathbf{3/4}}$ = 167 kilometers/sec.

Problem 4 – For what value of x is V(x) maximum?
Answer: Students can graph this function on a calculator. The maximum should occur near **x = 1.4** with a value V(x) = 217 km/sec.

Advanced students can use differential calculus and solve for x in the equation dV(x)/dx = 0.

$$\frac{dV(x)}{dx} = \frac{(350x)(3/4)(1+x^2)^{\frac{-1}{4}}(2x) - 350(1+x^2)^{\frac{3}{4}}}{(1+x^2)^{\frac{3}{2}}}$$

so after some algebra:

$$0 = 1 - \frac{3x^2}{2(1+x^2)}$$ so $2 + 2x^2 = 3x^2$ and $x = (2)^{1/2}$ = **1.414**

Problem 5 – For x=1 the physical distance is 10,000 light years. How many years does it take a star to complete one circular orbit at x=1.0 if 1 light year equals 9.5×10^{12} km, and there are 3.1×10^7 seconds in a year? Answer: For x=1 the physical distance is 10,000 light years or 9×10^{16} kilometers. The circumference of the orbit is $2 \pi R = 2 (3.141) (9.5 \times 10^{16}$ km) = 6.0×10^{17} kilometers. The speed is V(1) = 208 km/sec, so the time in seconds is $T = 6 \times 10^{17}$ kilometers / (208 km/sec) = 2.9×10^{15} seconds. Since there are 3.1×10^7 seconds/year, it will take **93 million years for a star to orbit once-around the center of M-101**.