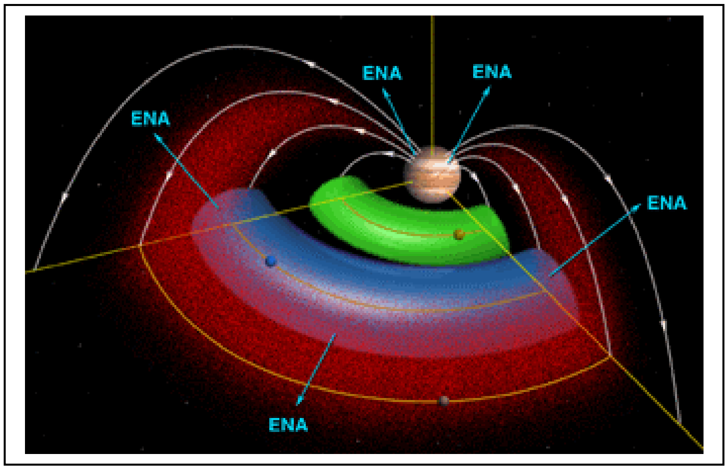
The Io Plasma Torus



The satellite of Jupiter, Io, is a volcanically active moon that ejects 1,000 kilograms of ionized gas into space every second. This gas forms a torus encircling Jupiter along the orbit of Io. We will estimate the total mass of this gas based on data from the NASA Cassini and Galileo spacecraft.

Image: Io plasma torus (Courtesy NASA/Cassini)

Problem 1 - Galileo measurements obtained in 2001 indicated that the density of neutral sodium atoms in the torus is about 35 atoms/cm^3. The spacecraft also determined that the inner boundary of the torus is at about 5 Rj, while the outer boundary is at about 8 Rj. (1 Rj = 71,300 km). A torus is defined by the radius of the ring from its center, R, and the radius of the circular cross section through the donut, r. What are the dimensions, in kilometers, of the Io torus based on the information provided by Galileo?

Problem 2 - Think of a torus as a curled up cylinder. What is the general formula for the volume of a torus with radii R and r?

Problem 3 - From the dimensions of the Io torus, what is the volume of the Io torus in cubic meters?

Problem 4 - From the density of sodium atoms in the torus, what is A) the total number of sodium atoms in the torus? B) If the mass of a sodium atom is 3.7×10^{-20} kilograms, what is the total mass of the Io torus in metric tons?

Calculus:

Problem 5 - Using the 'washer method' in integral calculus, derive the formula for the volume of a torus with a radius equal to R, and a cross-section defined by the formula $x^2 + y^2 = r^2$. The torus is formed by revolving the cross section about the Y axis.

Answer Key

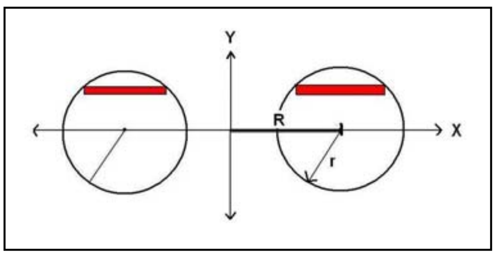
Problem 1 - The mid point between 5 Rj and 8 Rj is (8+5)/2 = 6.5 Rj so R = 6.5 Rj and r = 1.5 Rj. Then R = 6.5 x 71,300 so **R = 4.6 x 10^5 km**, and r = 1.5 x 71,300 so **r = 1.1 x 10^5 km**.

Problem 2 - The cross-section of the cylinder is πr^2, and the height of the cylinder is the circumference of the torus which equals 2 π R, so the volume is just V = (2 π R) x (π r^2) or **V = 2 $\pi^2 Rr^2$.**

Problem 3 - Volume = 2 π^2 (4.6 x 10^5 km) (1.1 x 10^5 km $)^2$ so **V = 1.1 x 10^{17} km^3.**

Problem 4 - A) 35 atoms/cm^3 x (100000 cm/1 km$)^3$ = 3.5 x 10^{16} atoms/km^3. Then number = density x volume so N = (3.5 x 10^{16} atoms/km^3) x (1.1 x 10^{17} km^3) , so **N = 3.9 x 10^{33} atoms**. B) The total mass is M = 3.9 x 10^{33} atoms x 3.7 x 10^{-20} kilograms/atom = 1.4 x 10^{14} kilograms. 1 metric ton = 1000 kilograms, so the total mass is **M = 100 billion tons**.

Advanced Math:



$$V = 8\pi R \int_0^r \left(r^2 - y^2 \right)^{1/2} dy$$

Recall that the volume of a washer is given by V = π (R(outer$)^2$ - R(inner$)^2$) x thickness. For the torus figure above, we see that the thickness is just dy. The distance from the center of the cross section to a point on the circumference is given by $r^2 = x^2 + y^2$. The width of the washer (the red volume element in the figure) is parallel to the X-axis, so we want to express its length in terms of y, so we get x = $(r^2 - y^2)^{1/2}$. The location of the outer radius is then given by R(outer) = R + $(r^2-y^2)^{1/2}$, and the inner radius by R(inner) = R - $(r^2 - y^2)^{1/2}$. We can now express the differential volume element of the washer by dV = π [$(R + (r^2-y^2)^{1/2})^2 - (R - (r^2 - y^2)^{1/2})^2$] dy. This simplifies to dV = π [4R $(r^2-y^2)^{1/2}$] dy or dV = 4 π R $(r^2-y^2)^{1/2}$ dy. The integral can immediately be formed from this ,with the limits y = 0 to y=r. Because the limits to y only span the upper half plane, we have to double this integral to get the additional volume in the lower half-plane. The required integral is shown above.

This integral can be solved by factoring out the r from within the square-root, then using the substitution U = y/r and dU = 1/r dy to get the integrand dV = 8 π R r^2 (1 - $U^2)^{1/2}$ dU. The integration limits now become U=0 to U=1. Since r and R are constants, this is an elementary integral with the solution V = 1/2 U $(1-U^2)^{1/2}$ + 1/2 arcsin(U). When this is evaluated from U=0 to U=1, we get

$$V = 8 \pi R r^2 \ [0 + 1/2 \arcsin(1)] - [0 + 1/2\arcsin(0)]$$
$$V = 8 \pi R r^2 \, 1/2 (\pi/2)$$
$$\mathbf{V = 2 \pi R r^2}$$