Exploring the Donut-shaped Van Allen Belts

The Van Allen belts were discovered in the late-1950s and resemble two donut-shaped clouds of protons (inner belt) and electrons (outer belt) with Earth at its center.

A donut is an example of a simple mathematical shape called a **torus** that is created by rotating a circle with a radius of \( r \), through a circular path with a radius of \( R \).

In terms of the variables \( r \) and \( R \), the formula for the volume of a torus is given by the rather scary-looking formula:

\[
V = 2\pi^2 R r^2
\]

**Problem 1** – What is the circumference of the circle with a radius of \( R \)?

**Problem 2** – What is the area of a circle with a radius of \( r \)?

**Problem 3** – If you dragged the area of the circle in Problem 2, along a distance equal to the circumference of the circle in Problem 2, what would be the formula for the volume that you swept out?

**Problem 4** – If the Van Allen belts can be approximated by a torus with \( r = 16,000 \) km, and \( R = 26,000 \) km, to two significant figures, what is the total volume of the Van Allen belts in cubic kilometers?

**Problem 5** – To two significant figures, how many spherical Earths can you fit in this volume if \( r = 6378 \) km?

Problem 1 – What is the circumference of the circle with a radius of R?
Answer: \[ C = 2 \pi R \]

Problem 2 – What is the area of a circle with a radius of r?
Answer: \[ A = \pi r^2 \]

Problem 3 – If you dragged the area of the circle in Problem 2, along a distance equal to the circumference of the circle in Problem 2, what would be the formula for the volume that you swept out?
Answer: Volume = Area x distance
\[ = (\pi r^2) \times (2 \pi R) \]
\[ = 2 \pi^2 R r^2 \]

Problem 4 – If the Van Allen belts can be approximated by a torus with r = 16,000 km, and R = 26,000 km, to two significant figures what is the total volume of the Van Allen belts in cubic kilometers?
Answer:
\[ r = 16000 \text{ km} \times (1000 \text{ m/1km}) = 16,000,000 \text{ meters} \]
\[ R = 26000 \text{ km} \times (1000 \text{ m/1km}) = 26,000,000 \text{ meters} \]
\[ V = 2 (3.14)^2 (2.6 \times 10^7) (1.6 \times 10^7)^2 \]
\[ = 1.3 \times 10^{23} \text{ meters}^3 \]

Problem 5 – To two significant figures, how many spherical Earths can you fit in this volume if \( r = 6378 \text{ km} \)?
Answer: \[ V = \frac{4}{3} \pi r^3 \]
\[ V = 1.33 (3.14) (6.378 \times 10^6 \text{ m})^3 \]
\[ V = 1.1 \times 10^{21} \text{ meters}^3 \]
So \[ 1.3 \times 10^{23} \text{ meters}^3 / 1.1 \times 10^{21} \text{ meters}^3 = 118 \] or \textbf{120 Earths!}