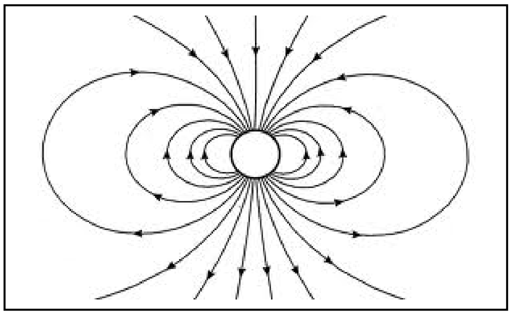
Mathematical Model of Magnetic Field Lines - III



Mathematically, every point in space near a magnet can be represented by a vector, B. Because the field exists in 3-dimensional space, it has three 'components'. The equations for the coordinates of B in 2-dimensions looks like this:

$$B_r = -\frac{2M\sin\theta}{r^3} \qquad B_\theta = \frac{M\cos\theta}{r^3}$$

It is convenient to graph a magnetic field on a 2-dimensional piece of paper to show its shape. The lines that are drawn are called 'magnetic field lines', and if you placed a compass at a particular point on the field line, the direction of the line points to 'north' or 'south'.

The slope of the magnetic field at any point (R,θ) is defined by $\frac{B_\theta}{B_r}$.

From calculus, in a polar coordinate system, the slope of a line is defined by $\frac{rd\theta}{dr}$.

Problem 1 – What is the differential equation that relates $\frac{B_\theta}{B_r}$ to $\frac{rd\theta}{dr}$?

Problem 2 – Integrate your answer to Problem 1 to find the polar coordinate equation of a magnetic field line.

Problem 1 -

$$\frac{rd\theta}{dr} = \frac{M\cos\theta}{-2M\sin\theta} \quad \text{so} \quad \frac{rd\theta}{dr} = -\frac{\cos\theta}{2\sin\theta}$$

Problem 2 -

Rearrange the terms into two integrands: $$\frac{dr}{r} = -\frac{2\sin\theta}{\cos\theta}d\theta$$

The integrals become $$\int\frac{dr}{r} = -2\int\frac{\sin\theta}{\cos\theta}d\theta$$

These are both logarithmic integrals that yield the solution:

$$ln(r) \ + \ \mathrm{C} \ = 2\ ln(\cos\theta) \ + \ \mathrm{C}$$

R_0 is the distance from the center of the magnetic field to the point where the field line crosses the equatorial plane of the magnet at θ=0. Each field line is specified by a unique crossing point distance. In other words, the constants of integration are specified by the condition that r = r_0 for θ = 0, which then gives us the final form of the equation:

$$r = r_0\cos^2\theta$$