



An important quantity in planetary exploration is the distance to the horizon. This will, naturally, depend on the diameter of the planet (or asteroid!) and the height of the observer above the ground.

Another application of this geometry is in determining the height of a transmission antenna on the Moon in order to insure proper reception out to a specified distance.

Teachers: Problems 1-4 can be successfully accomplished by algebra students. Problems 5 and 6 require a knowledge of derivatives and can be assigned to calculus students after they have completed Problem 1 and 2.

Problem 1: If the radius of the planet is given by R , and the height above the surface is given by h , use the figure above to derive the formula for the line-of-sight (LOS) horizon distance, d , to the horizon tangent point.

Problem 2: Derive the distance along the planet, L , to the tangent point.

Problem 3: For a typical human height of 2 meters, what is the horizon distance on A) Earth ($R=6,378$ km); B) The Moon ($1,738$ km)

Problem 4: A radio station has an antenna tower 50 meters tall. What is the maximum line-of-sight (LOS) reception distance in the Moon?

Problem 5: What is the rate of change of the lunar LOS radius, d , for each additional meter of antenna height in Problem 4?

Problem 6: What is the rate-of-change of the distance to the lunar radio tower, L , at the LOS position in Problem 4?

Answer Key:

Problem 1: By the Pythagorean Theorem $d^2 = (R+h)^2 - R^2$
 so $d = (R^2 + 2Rh + h^2 - R^2)^{1/2}$
 and so $d = (h^2 + 2Rh)^{1/2}$

Problem 2: Derive the distance along the planet, l , to the tangent point. From the diagram,
 $\cos(\beta) = R/(R+h)$ and so $L = R \arccos(R/(R+h))$

Problem 3: Use the equation from Problem 1. A) For Earth, $R=6378$ km and $h=2$ meters so
 $d = ((2 \text{ meters})^2 + 2 \times 2 \text{ meters} \times 6.378 \times 10^6 \text{ meters})^{1/2} = 5051$ meters or 5.1 kilometers.
 B) For the Moon, $R=1,738$ km so $d = 2.6$ kilometers

Problem 4: $h = 50$ meters, $R=1,738$ km so $d = 13,183$ meters or **13.2 kilometers**.

Problem 5: Use the chain rule to take the derivative with respect to h of the equation for d in Problem 1. Evaluate dd/dh at $h=50$ meters for $R=1,738$ km.

Let $U = h^2 + 2Rh$ then $d = U^{1/2}$ so $dU/dh = (dd/dU) (dU/dh)$

Then $dd/dh = +1/2 U^{-1/2}$

$dU/dh = +1/2 (2h + 2R) (h^2 + 2Rh)^{-1/2}$

For $h=50$ meters and $R = 1,738$ km,

$$\begin{aligned} dd/dh &= +0.5 \times (100 + 3476000) (2500 + 2 \times 50 \times 1738000)^{-1/2} \\ &= \mathbf{+131.8 \text{ meters in LOS distance per meter of height.}} \end{aligned}$$

Problem 6: Let $U = R/(R+h)$, then $L = R \cos^{-1}(U)$.

By the chain rule $dL/dh = (dL/dU) \times (dU/dh)$.

Since $dL/dU = R \times (-1)(1 - u^2)^{-1/2}$ and $dU/dh = R \times (-1) \times (R+h)^{-2}$ then

$$dL/dh = R^2 (R+h)^{-2} (R+h)^{1/2} / ((R+h)^2 - R^2)^{1/2}$$

$$dL/dh = R^2 (R+h)^{-1} (h^2 + 2Rh)^{-1/2}$$

Since $R \gg h$, $dL/dh = R/(2Rh)^{1/2}$

Evaluating this for $R = 1,738$ km and $h = 50$ meters gives $dL/dh = \mathbf{+131.8 \text{ meters per kilometer}}$.