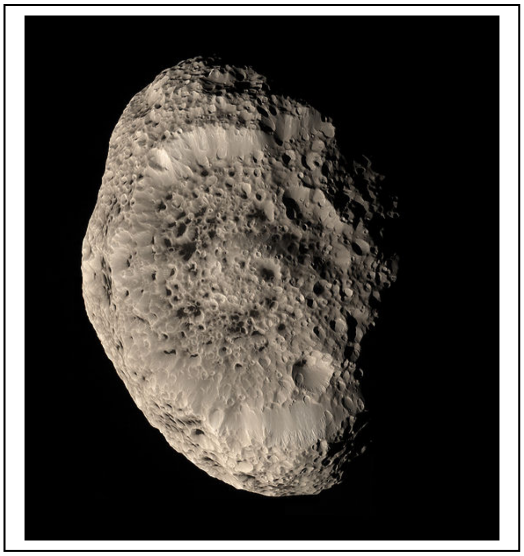
From Asteroids to Planets



Planets are built in several stages. Dust grains grow to large rocks in a million years, then rocks accumulate to form asteroids in a few years or so. The third stage combines kilometer-wide asteroids to make rocky planets. A simple model of this process can tell us about how long it takes to 'grow' a planet by accumulating asteroid-sized bodies through collisions. Saturn's moon Hyperion (see image) is 300 km across and is an example of a 'small' planet-sized body called a planetoid.

Problem 1 – Assume that the forming planet is spherical with a density of 3 gm/cc, a radius R, and a mass M. If the radius is a function of time, R(t), what is the equation for the mass of the planet as a function of time, M(t)?

Problem 2 – The planet grows by absorbing incoming asteroids that have an average mass of 10^{15} grams and a density of N asteroids per cubic centimeter in the cloud. The asteroids collide with the surface of the forming planet at a speed of V cm/sec, what is the equation that gives the rate of growth of the planet's mass in time (dM/dt)?

Problem 3 – From your answer to Problem 1 and 2, re-write dM/dt in terms of M not R.

Problem 4 – Integrate your answer to Problem 3 so determine M(t).

Problem 5 – What is the mass of the planet when it reaches a diameter of 5000 kilometers if its density is 3 grams/cc?

Problem 6 – The planetoid begins at t=0 with a mass of m= 2 x 10^{15} grams. The cloud density N = 1.0 x 10^{-24} asteroids/cc (1 asteroid per 1000 cubic kilometers), and the speed of the asteroids striking the planet, without destroying the planet, is V=1 kilometer/sec. How many years will the growth phase have to last for the planet to reach a diameter of 5000 kilometers?

Problem 1 – Answer: Because mass = density x volume, we have
M = 4/3 π R^3 ρ and so **M(t) = 4/3 π ρ $R(t)^3$**

Problem 2 –**Answer:** The change in the mass, dM, occurs as a quantity of asteroids land on the surface area of the planet per unit time, dt. The amount is proportional to the surface area of the planet, since the more surface area the planet has, the more asteroids will be absorbed. Also, the rate at which asteroid mass is brought to the surface of the forming planet is proportional to the product of the asteroid density in the planetary nebula, times the speed of the asteroids landing on the surface of the planet. This leads to m x N x V where m is in grams per dust grain, N is in asteroids per cubic centimeter, and V is in centimeters/sec. The product of all three factors has the units of grams per square centimeter per second. The product of (m x N x V) with the surface area of the planet, will then have the units of grams/sec representing the rate at which the planet mass is growing. The full formula for the growth of the planet mass is then
dM/dt = 4 p R^2 m N V

Problem 3 – Answer: From Problem 1 we see that R(t) = $(3\,M(t)/\,4\,\pi\,\rho)^{1/3}$. Then substituting into dM/dt we have dM/dt = 4 π m N V $(3\,M(t)/4\,\pi\,\rho)^{2/3}$ so

$$\frac{dM(t)}{dt} = 4\pi mNV\left(\frac{3}{4\pi\rho}\right)^{\frac{2}{3}} M(t)^{\frac{2}{3}}$$

Problem 4 –Answer: Re-write the differentials and move M(t) to the side with dM to get the integrand $M(t)^{-2/3}$ dM = 4 π m N V $(3/4\,\pi\,\rho)^{2/3}$ dt Then integrate both sides to get:
3 $M(t)^{1/3}$ = 4 π m N V $(3/4\,\pi\,\rho)^{2/3}$ t + c. Solve for M(t) to get the final equation for M(t), and remember to include the integration constant, c:

$$M(t) = \left[\frac{4}{3}\pi mNV\left(\frac{3}{4\pi\rho}\right)^{\frac{2}{3}} t + c\right]^3$$

Problem 5 – What is the mass of the planet when it reaches a diameter of 5000 kilometers if its density is 3 grams/cc? Answer M = 4/3 π $(2.5 \times 10^8\ \text{cm})^3$ x 3.0 gm/cc = **2.0×10^{26} grams.**

Problem 6 – The planetoid begins at t=0 with a mass of 1 asteroid, m = 2.0×10^{15} grams. The cloud density N = 1.0×10^{-24} asteroids/cc (This equals 1 asteroid per 1000 cubic kilometers) and the speed of the asteroids striking the planet, without destroying the planet, is V=1 kilometer/sec. How many years will the growth phase have to last for the planet to reach a diameter of 5000 kilometers? Answer: For t = 0, M(0) = m so the constant of integration is c = $m^{1/3}$ so c = $(2.0 \times 10^{15}\ \text{g})^{1/3}$ = 1.3×10^5.
Then M(t) = $(4/3\ (3.14)\ (2 \times 10^{15})(1.0 \times 10^{-24})(100{,}000)(3/(4(3.14)\ (3.0)))^{2/3}\ t + 1.3 \times 10^5)^3$
M(t) = $(0.00015\ t + 1.3 \times 10^5)^3$

To get M(t) = 2.0×10^{26} grams will take t = 3.9×10^{12} seconds or about **126,000 years!**