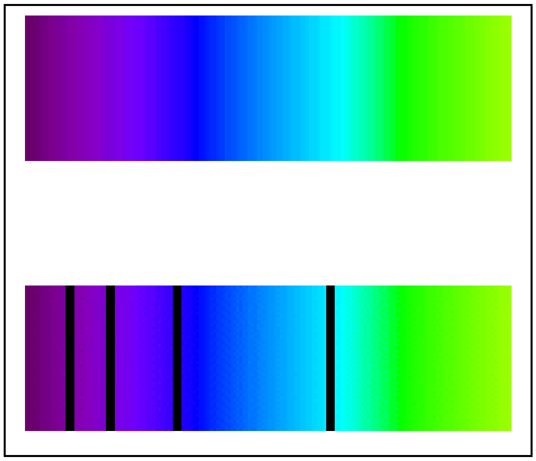


The spectrum of the element hydrogen is shown to the left. The dark 'spectral' lines that make-up the fingerprint of hydrogen only exist at specific wavelengths. This allows astronomers to identify this gas in many different bodies in the universe.

The energy corresponding to each line follows a simple mathematical series because at the atomic-scale, energy comes in the form of specific packets of light energy called quanta.

The Lyman Series of hydrogen lines is determined by the term relation:

$$E_n = 13.7\left(1 - \frac{1}{n^2}\right) \text{ electron Volts}$$

where n is the energy level, which is a positive integer from 1 to infinity, and E_n is the energy in electron Volts (eV) between level n and then lowest 'ground state' level n=1. E_n determines the energy of the light emitted by the hydrogen atom when an electron loses energy by making a jump from level n to the ground state level.

Problem 1 – Compute the energy in eV of the first six spectral lines for the hydrogen atom using E_n.

Problem 2 – Suppose an electron jumped from an energy level of n=7 to a lower level where n = 3. What is the absolute magnitude of the energy difference between level n = 3 and level n = 7?

Problem 1 – Compute the energy in eV of the first six spectral lines for the hydrogen atom using the formula for E_n.

Answer: Example: E_2 = 13.7 (1-1/4) = 13.7 x ¾ = 10.3 eV.

E_2 = 10.3 eV

E_3 = 12.2 eV

E_4 = 12.8 eV

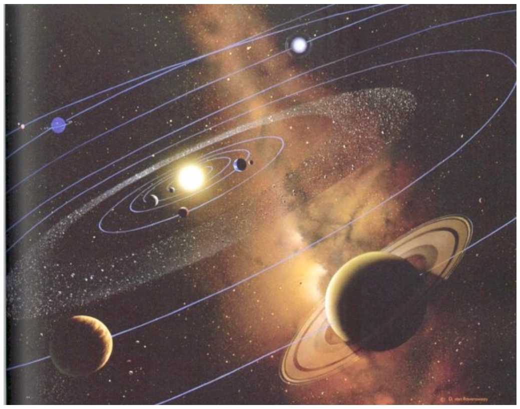
E_5 = 13.2 eV

E_6 = 13.3 eV

E_7 = 13.4 eV

Problem 2 – Suppose an electron jumped from an energy level of n=7 to a lower level where n = 3. What is the absolute magnitude of the energy difference between level n = 3 and level n = 7?

Answer: E_3 = 12.2 eV and E_7 = 13.4 eV so **E_7-E_3 = 1.2 eV**



Long before the planet Uranus was discovered in 1781, it was thought that their distances from the sun might have to do with some mathematical relationship. Many proposed distance laws were popular as early as 1715.

Among the many proposals was one developed by Johann Titius in 1766 and Johann Bode 1772 who independently found a simple series progression that matched up with the planetary distances rather remarkably.

Problem 1 – Compute the first eight terms, n=0 through n=7, in the Titius-Bode Law whose terms are defied by $Dn = 0.4 + 0.3*2^{n}$ where n is the planet number beginning with Venus (n=0). For example, for Neptune, N = 7 so Dn = 0.4 + 0.3 (128) = 38.8 AU.

Problem 2 – A similar series can be determined for the satellites of Jupiter, called Dermott's Law, for which each term is defined by $Tn = 0.44\ (2.03)^{n}$ and gives the orbit period of the satellite in days What are the orbital periods for the first six satellites of Jupiter?

Problem 1 – Compute the first eight terms in the Titius-Bode Law whose terms are defied by $Dn = 0.4 + 0.3*2^n$ where n is the planet number beginning with Venus (n=0). For example, for Neptune, N = 7 so Dn = 0.4 + 0.3 (128) = 38.8 AU.

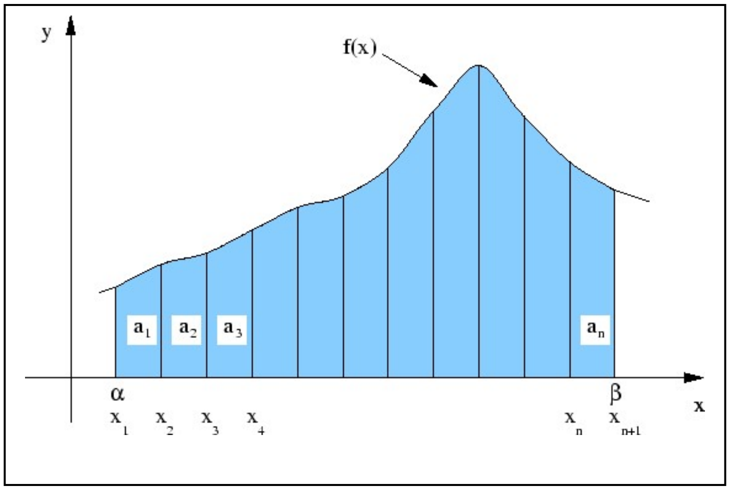
Answer: For n = 0, 1, 2, 3, 4, 5, 6 and 7 the distances are

Venus:	**d0 = 0.7**	actual planet distance = 0.69
Earth:	**d1 = 1.0**	actual planet distance = 1.0
Mars:	**d2 = 1.6**	actual planet distance = 1.52
Ceres:	**d3 = 2.8**	actual planet distance = 2.77
Jupiter:	**d4 = 5.2**	actual planet distance = 5.2
Saturn:	**d5 = 10.0**	actual planet distance = 9.54
Uranus:	**d6 = 19.6**	actual planet distance = 19.2
Neptune:	**d7 = 38.8**	actual planet distance = 30.06

Note: Ceres is a large asteroid not a planet.

Problem 2 – A similar series can be determined for the satellites of Jupiter, called Dermott's Law, for which each term is defined by $Tn = 0.44 (2.03)^n$ and gives the orbit period of the satellite in days What are the orbital periods for the first six satellites of Jupiter?

Answer: **T0 = 0.44 days**
T1 = 0.89 days
T2 = 1.81 days
T3 = 3.68 days
T4 = 7.47 days
T5 = 15.17 days



Arithmetic series appear in many different ways in astronomy and space science. The most common is in determining the areas under curves.

For example, an arithmetic series is formed from the addition of the rectangular areas $\mathbf{a_n}$ in the figure to the left.

Imagine a car traveling at a speed of 11 meters/sec and wants to accelerate smoothly to 22 meters/sec to enter a freeway. As it accelerates, its speed changes from 11 m/sec at the first second, to 12 m/sec after the second second and 13 m/sec after the third second and so on.

Problem 1 – What is the general formula for the Nth term in this series for V_n where the first term in the series, V_1 = 11 m/sec?

Problem 2 – What is the value of the term V_8 in meters/sec?

Problem 3 – For what value of N will V_n = 22 meters/sec?

Problem 4 – What is the sum, S_{12}, of the first 12 terms in the series?

Problem 5 – If the distance traveled is given by D = S_{12} x T where T is the time interval between each term in the series, how far did the car travel in order to reach 22 meters/sec?

Problem 1 – What is the general formula for the Nth term in this series for V_n where V_1 = 11 m/sec?

Answer: **$V_n = 10 + 1.0n$**

Problem 2 – What is the value of the term V_8 in meters/sec?

Answer: $V_8 = 10 + 1.0 *(8)$ so **V_8 = 18.0 m/sec**

Problem 3 – For what value of N will V_n = 22 meters/sec?

Answer: 22 = 10 + 1.0 N so **N = 12**

Problem 4 – What is the sum, S_{12}, of the first 12 terms in the series?

Answer:
The first 12 terms in the series are:
11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22

The sum of an arithmetic series is given by $S_n = n (a_1 + a_n)/2$
So for n = 12, $a_1 = 11$ and $a_{12} = 22$ we have **S_{12} = 211 m/sec**.

Problem 5 – If the distance traveled is given by $D = S_{12} \times T$ where T is the time interval between each term in the series, how far did the car travel in order to reach 22 meters/sec?

Answer: In the series for V_n, the time interval between terms is 1.0 seconds. Since S_{12} = 211 m/sec we have D = 211 m/sec x 1.0 sec so **D = 211 meters**.

The $150 million Deep Space 1 spacecraft launched on October 22, 1998 used an ion engine to travel from Earth to the Comet Borrelly. It arrived on September 22, 2001.

By ejecting a constant stream of xenon atoms into space, at speeds of thousands of kilometers per second, the new ion engine could run continuously for months. This allowed the spacecraft to accelerate to speeds that eventually could exceed the fastest rocket-powered spacecraft.

Problem 1 – The Deep Space 1 ion engine produced a constant acceleration, starting from a speed of 44,000 km/hr, reaching a speed of 56,060 km/hr as it passed the comet 36 months later. The series representing the monthly average speed of the spacecraft can be approximated by a series based upon its first 7 months of operation given by:

n	1	2	3	4	5	6	7
V_n	44,000	44,335	44,670	45,005	45,340	45,675	46,010

What is the general formula for Vn?

Problem 2 – Suppose the Deep Space I ion engine could be left on for 30 years! What would be the speed of the spacecraft at that time?

Problem 3 – The sum of an arithmetic series is given by $S_n = n (a_1 + a_n)/2$. What is the sum, S_{36}, of the first 36 terms of this series?

Problem 4 – The total distance traveled is given by $D = S_{36}$ x T where T is the time between series terms in hours. How far did the Deep Space 1 spacecraft travel in reaching Comet Borrelly?

Problem 1 – The Deep Space 1 ion engine produced a constant acceleration, starting from a speed of 44,000 km/hr, reaching a speed of 56,060 km/hr as it passed the comet 36 months later. The series representing the monthly average speed of the spacecraft can be approximated by a series based upon its first 7 months of operation given by:

n	1	2	3	4	5	6	7
V_n	44,000	44,335	44,670	45,005	45,340	45,675	46,010

What is the general formula for Vn?

Answer: **Vn = 44,000 + 335(n-1)**

Problem 2 – Suppose the Deep Space I ion engine could be left on for 30 years! What would be the speed of the spacecraft at that time?

Answer: 30 years = 30 x 12 = 360 months so the relevant term in the series is V_{360} which has a value of V360 = 44,000 + 335x(360-1) so **V_{360} = 164,265 kilometers/hour**.

Problem 3 – What is the sum, S_{36}, of the first 36 terms of this series?

Answer: V_{36} = 44,000 + 11,725 = 55,725 km/hour. Then S_{36} = 36 (44000 + 55,725)/2 so **S_{36} = 1,795,050 kilometers/hour**.

Problem 4 – The total distance traveled is given by D = S_{36} x T where T is the time between series terms in hours. How far did the Deep Space 1 spacecraft travel in reaching Comet Borrelly if there are 30 days in a month?

Answer: The time between each series term is 1 month which equals 30 days x 24hours/day = 720 hours. The total distance traveled is then

D = 1,795,050 km/hr x 720 hours

D = 1,292,436,000 kilometers.

Note, this path was a spiral curve between the orbit of Earth and the comet. During this time, it traveled a distance equal to 8.7 times the distance from the Sun to Earth!

When light passes through a dust cloud, it decreases in intensity. This decrease can be modeled by a geometric series where each term represents the amount of light lost from the original beam of light entering the cloud.

The image to the left shows the dark cloud called Barnard 68 photographed by astronomers at the ESO, Very Large Telescope observatory. The dust cloud is about 500 light years from Earth and about 1 light year across.

Problem 1 – A dust cloud causes starlight to be diminished by 1% in intensity for each 100 billion kilometers that it travels through the cloud. If the initial starlight has a brightness of B_1 = 350 lumens, what is the geometric series that defines its brightness?

Problem 2 – What are the first 8 terms in this series for the brightness of the light?

Problem 3 - How far would the light have to penetrate the cloud before it loses 50% of its original intensity?

Problem 1 – A dust cloud causes starlight to be diminished by 1% in intensity for each 100 billion kilometers that it travels through the cloud. If the initial starlight has a brightness of B_1 = 350 lumens, what is the geometric series that defines its brightness?

Answer: B_1 = 350 and r = 0.01 so if each term represents a step of 100 billion km in distance, the series is **$B_n = 350 (0.99)^{n-1}$**

Problem 2 – What are the first 8 terms in this series for the brightness of the light?

Answer: Calculate B_n for n = 1, 2, 3, 4, 5, 6, 7, 8

N	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Bn	**350**	**346**	**343**	**340**	**336**	**333**	**330**	**326**

Problem 3 - How far would the light have to penetrate the cloud before it loses 50% of its original intensity?

Answer: Find the term number for which B_n = 0.5*350 = 175. Then
$175 = 350 (0.99)^{n-1}$ solve for n using logarithms:

Log(175) = Log(350) + (n-1) log(0.99)

so n-1 = (log(175) – Log(350))/log(0.99)
and so n-1 = 68.96
or **n = 68**.

Since the distance between each term is 100 billion km, the penetration distance to half-intensity will be 68 x 100 billion km = **6.8 trillion kilometers**.

Note: 1 light year = 9.3 trillion km, the distance is just under 1 light year.

The star field shown above was photographed by NASA's WISE satellite and shows thousands of stars, and represents an area of the sky about the size of the full moon. Notice that the stars come in many different brightnesses. Astronomers describe the distribution of stars in the sky by counting the number in various brightness bins.

Suppose that after counting the stars in this way, an astronomer determines that the number of the can be modeled by an infinite geometric series: $B_m = 100\ a^{m-1}$ where **a** is a scaling number between 1/3 and 1/2. The series term index, m, is related to the apparent magnitude of the stars in the star field and ranges from m:[1 to +infinity].

Problem 1 –What are the first 7 terms in this series for a=0.398?

Problem 2 - What is the sum of the geometric series, B_m , for A) a=0.333? B) a=0.398? C) a=0.5?

Problem 1 – Suppose that the brightness of this field can be approximately given by the geometric series $B_m = 100\ a^{m-1}$ where **a** is a number between 1/2 and 1/3. The series term index, m, is related to the apparent magnitude of the stars in the star field and ranges from m:[1 to +infinity]. What are the first 7 terms in this series for a=0.398?

Answer: a = 0.398 then:

m	1	2	3	4	5	6	7
Bm	100	40	16	6.3	2.5	1.0	0.4

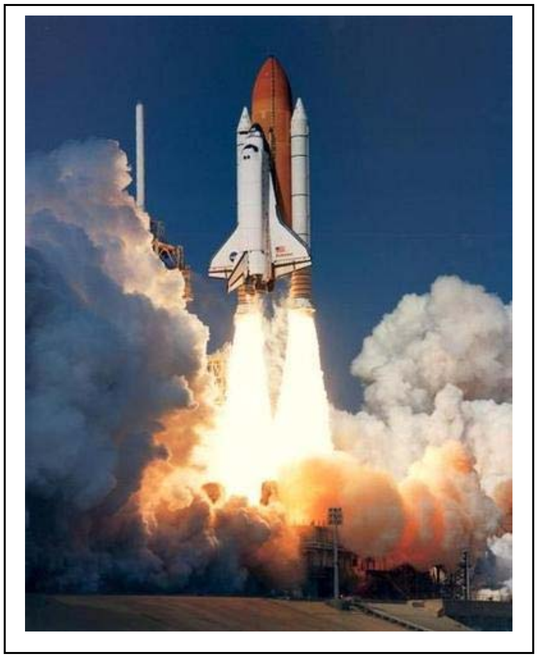
Problem 2 - What is the sum of this geometric series for A) a=0.333? B) a=0.398? C) a=0.5?

Answer: A) The common ratio is 0.333 and the first term has a value of B_1 = 100, so B = 100 / (1-0.333) and so **B= 150.**

B) The common ratio is 0.398 and the first term has a value of B_1 = 100, so B = 100 / (1-0.398) and so **B= 166.**

C) The common ratio is 0.50 and the first term has a value of B_1 = 100, so B = 100 / (1-0.50) and so **B= 200.**

Note: Normally, star counts are always referred to a specific magnitude system since the brightness of stars at different wavelengths varies.



Rockets work by throwing mass out their ends to produce 'thrust', which moves the rocket forward.

As fuel mass leaves the rocket, the mass of the rocket decreases and so the speed of the rocket steadily increases as the rocket becomes lighter and lighter.

An interesting feature of all rockets that work in this way is that the maximum attainable speed of the rocket is determined by the Rocket Equation, which can be understood by using an infinite geometric series.

Problem 1 – The Rocket Equation can be approximated by the series $V = V_1 a^{n-1}$, where **a** is a quantity that varies with the mass ratio of the surviving payload mass, m, to the initial rocket mass, M. For a rocket in which the payload mass is 10% of the total fueled rocket mass, a = 0.56. What are the first 5 terms in the equation for the rocket speed, V, if the exhaust speed is V_1 = 2,500 meters/sec?

Problem 2 – The partial sums of the series, S_1, S_2, S_3, …, reflect the fact that, as the rocket burns fuel, the mass of the rocket decreases, so the speed will increase. For example, after two seconds, the second time interval, $S_2 = V_1 + V_2$ = 2,500 + 1,400 = 3,900 m/sec. After three seconds, the speed is S_3 = 2,500 + 1,400 + 784 = 4,684 m/sec etc. What is the speed of the rocket after A) 15 seconds? B) 35 seconds?

Problem 3 – The maximum speed of the payload is given by the limit to the sum of the series for V. What is the sum of this infinite series for V in meters/sec?

Problem 1 – The Rocket Equation can be approximated by the series $V = V_0 \, a^n$, where **a** is a quantity that varies with the mass ratio of the surviving payload mass, m, to the initial rocket mass, M. For a rocket in which the payload mass is 10% of the total fueled rocket mass, a = 0.56. What are the first 5 terms in the equation for the rocket speed, V, if the exhaust speed is 2,500 meters/sec?

Answer: $V_1 = 2500\,(0.56)^0 = 2{,}500$ m/sec
$V_2 = 2500\,(0.56)^1 = 1{,}400$ m/sec
$V_3 = 2500\,(0.56)^2 = 784$ m/sec
$V_4 = 2500\,(0.56)^3 = 439$ m/sec
$V_5 = 2500\,(0.56)^4 = 246$ m/sec

So the sequence is V=2,500 + 1,400 + 784 + 439 + 246 + …

Problem 2 – The partial sums of the series, S_1, S_2, S_3, …, reflect the fact that, as the rocket burns fuel, the mass of the rocket decreases, so the speed will increase. For example, after two seconds, the second time interval, $S_2 = V_1 + V_2$ = 2,500 + 1,400 = 3,900 m/sec. After three seconds, the speed is S_3 = 2,500 + 1,400 + 784 = 4,684 m/sec etc. What is the speed of the rocket after A) 15 seconds? B) 35 seconds?

Answer: A) Recall that the sum of a geometric series is given by $S_n = a(1-r^n)/(1 - r)$
So A) r = 0.56, a = 2500, n = 15 and so
$S_{15} = 2500\,(1-(0.56)^{15})/(1 - 0.56)$
S = **5,681 meters/sec**.
B) n = 35 so $S_{35} = 2500\,(1-(0.56)^{35})/(1-0.56)$
S = **5,682 meters/sec**.

Problem 3 – The maximum speed of the payload is given by the limit to the sum of the series for V. What is the sum of this infinite series for V in meters/sec?

Answer: S = a/(1-r) = 2500/(1-0.56) = **5,682 meters/sec**.

Note: This speed is equal to **20,500 km/hour**.

Recursive Rules for Sequences

This spherical propellant tank is an important component of testing for the Altair lunar lander, an integral part of NASA's Constellation Program. It will be filled with liquid methane and extensively tested in a simulated lunar thermal environment to determine how liquid methane would react to being stored on the moon.

The volume of a sphere is a mathematical quantity that can be extended to spaces with different numbers of dimensions.

The mathematical formula for the volume of a sphere in a space of N dimensions is given by the recursion relation

$$V(N) = \frac{2\pi R^2}{N} V(N-2)$$

For example, for 3-dimensional space, N = 3 and since from the table to the left, V(N-2) = V(1) = 2R, we have the usual formula

$$V(3) = \frac{4}{3}\pi R^3$$

Dimension	Formula	Volume
0	1	1.00
1	2R	2.00
2	πR^2	3.14
3	$\frac{4}{3}\pi R^3$	4.19
4		
5		
6		
7		
8		
9		
10		

Problem 1 - Calculate the volume formula for 'hyper-spheres' of dimension 4 through 10 and fill-in the second column in the table.

Problem 2 - Evaluate each formula for the volume of a sphere with a radius of R=1.00 and enter the answer in column 3.

Problem 3 - Create a graph that shows V(N) versus N. For what dimension of space, N, is the volume of a hypersphere its maximum possible value?

Problem 4 - As N increases without limit, what is the end behavior of the volume of an N-dimensional hypersphere?

Dimension	Formula	Volume
0	1	1.00
1	2R	2.00
2	πR^2	3.14
3	$\frac{4}{3}\pi R^3$	4.19
4	$\frac{\pi^2 R^4}{2}$	4.93
5	$\frac{8\pi^2 R^5}{15}$	5.26
6	$\frac{\pi^3 R^6}{6}$	5.16
7	$\frac{16\pi^3 R^7}{105}$	4.72
8	$\frac{\pi^4 R^8}{24}$	4.06
9	$\frac{32\pi^4 R^9}{945}$	3.30
10	$\frac{\pi^5 R^{10}}{120}$	2.55

Problem 1 - Answer for N=4:

$$V(4) = \frac{2\pi R^2}{4} V(4-2)$$

$$V(4) = \frac{2\pi R^2}{4} V(2)$$

$$V(4) = \frac{2\pi R^2}{4} (\pi R^2)$$

$$V(4) = \frac{\pi^2 R^4}{2}$$

Problem 2 - Answer for N=4:
$V(4) = (0.5)(3.141)^2 = 4.93$.

Problem 3 - The graph to the left shows that the maximum hypersphere volume occurs for spheres in the fifth dimension (N=5). Additional points have been calculated for N=11-20 to better illustrate the trend.

Problem 4 - In the limit for spaces with very large dimensions, the hypersphere volume approaches zero!