Spherical tanks are found in many different situations, from the storage of cryogenic liquids, to fuel tanks. Under the influence of gravity, or acceleration, the liquid will settle in a way such that it fills the interior of the tank up to a height, \( h \). We would like to know how full the tank is by measuring \( h \) and relating it to the remaining volume of the liquid. A sensor can then be designed to measure where the surface of the liquid is, and from this derive \( h \).

**Problem 1** - Slice the fluid into a series of vertically stacked disks with a radius \( r(h) \) and a thickness \( dh \). What is the general formula for the radius of each disk?

**Problem 2** - Set up the integral for the volume of the fluid and solve the integral.

**Problem 3** - Assume that fluid is being withdrawn from the tank at a fixed rate \( dV/dt = -F \). What is the equation for the change in the height of the fluid volume with respect to time? A) Solve for the limits \( h<<R \) and \( h>>R \). B) Solve graphically for \( R=1 \) meter, \( F=100 \text{ cm}^3/\text{min} \). (Hint: select values for \( h \) and solve for \( t \)).

Problem 1 - \[ r(h)^2 = R^2 - (R-h)^2 \] so \( r(h)^2 = 2Rh - h^2 \)

Problem 2 - The integrand will be \( \pi (2Rh - h^2) \, dh \) and the solution is \( \pi Rh^2 - \frac{1}{3} \pi h^3 \)

Problem 3 -

\[
\frac{dV}{dt} = 2 \pi R h \frac{dh}{dt} \quad \text{so} \quad \frac{dV}{dt} = \left(2 \pi R h - \pi h^2\right) \frac{dh}{dt} = -F
\]

Then

\[
\frac{dh}{dt} = \frac{-F}{2\pi R h - \pi h^2}
\]

The integrands become: \( (2 \pi R h - \pi h^2) \, dh = -F \, dt \). This can be integrated from \( t=0 \) to \( t=T \) to obtain \( \pi Rh^2 - \frac{1}{3} \pi h^3 = -FT \) and simplified to get

\[ h^3 - 3Rh^2 - \frac{(3FT)}{\pi} = 0 \]

We would normally like to invert this equation to get \( h(T) \), but cubic equations of the form \( x^3 - \alpha x^2 + \beta = 0 \) cannot be solved analytically. We can solve it for two limiting cases. Case 1 for a tank nearly empty where \( h \ll R \). This yields \( h(T) = \left(\frac{FT}{R}\right)^{1/2} \) Case 2 is for a tank nearly full so that \( h \gg R \), and we get \( h^3 = 3FT/\pi \) and \( h(T) = \left(\frac{3FT}{\pi}\right)^{1/3} \). The full solution for \( h(T) \) can be solved graphically. Since \( R \) is a constant, we can select a new variable \( U = h/R \) and re-write the equation in terms of the magnitude of \( h \) relative to the radius of the tank.

\[ U^3 - 3U^2 = \left(\frac{3FT}{\pi}\right)R^2 \] and plot this for selected combinations of \( (U,T) \) where time, \( T \), is the dependent variable. The solution below is for \( F = 100 \, \text{cm}^3/\text{minute} \), \( R = 1 \, \text{meter} \), with the intervals in \( h \) spaced 10 cm. The plot was generated using an Excel spreadsheet.

[Graph]