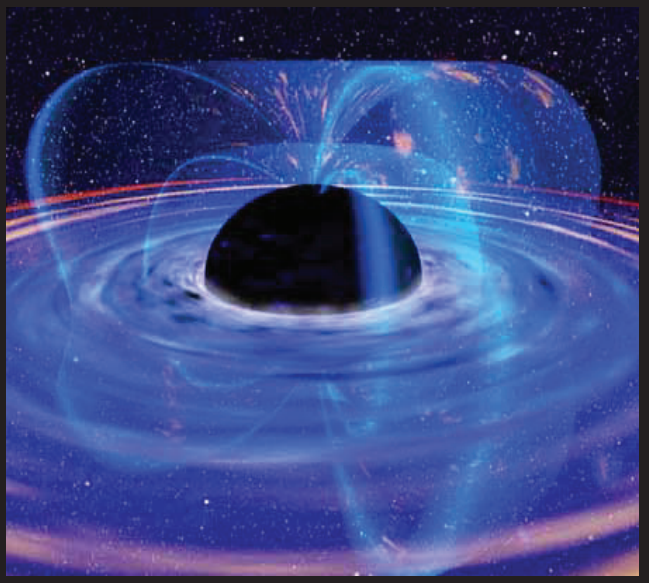
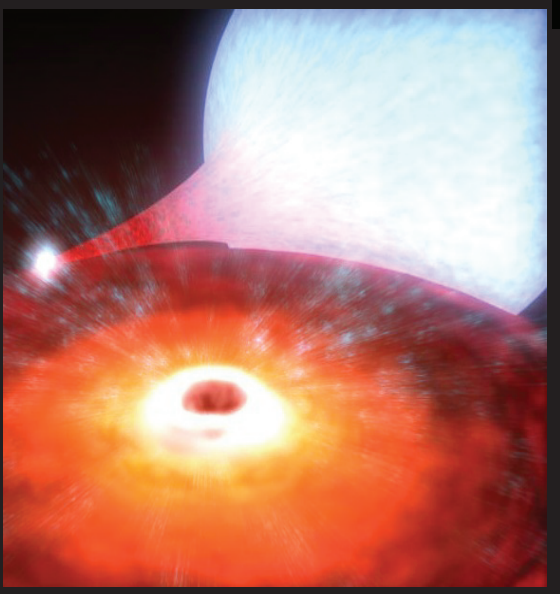
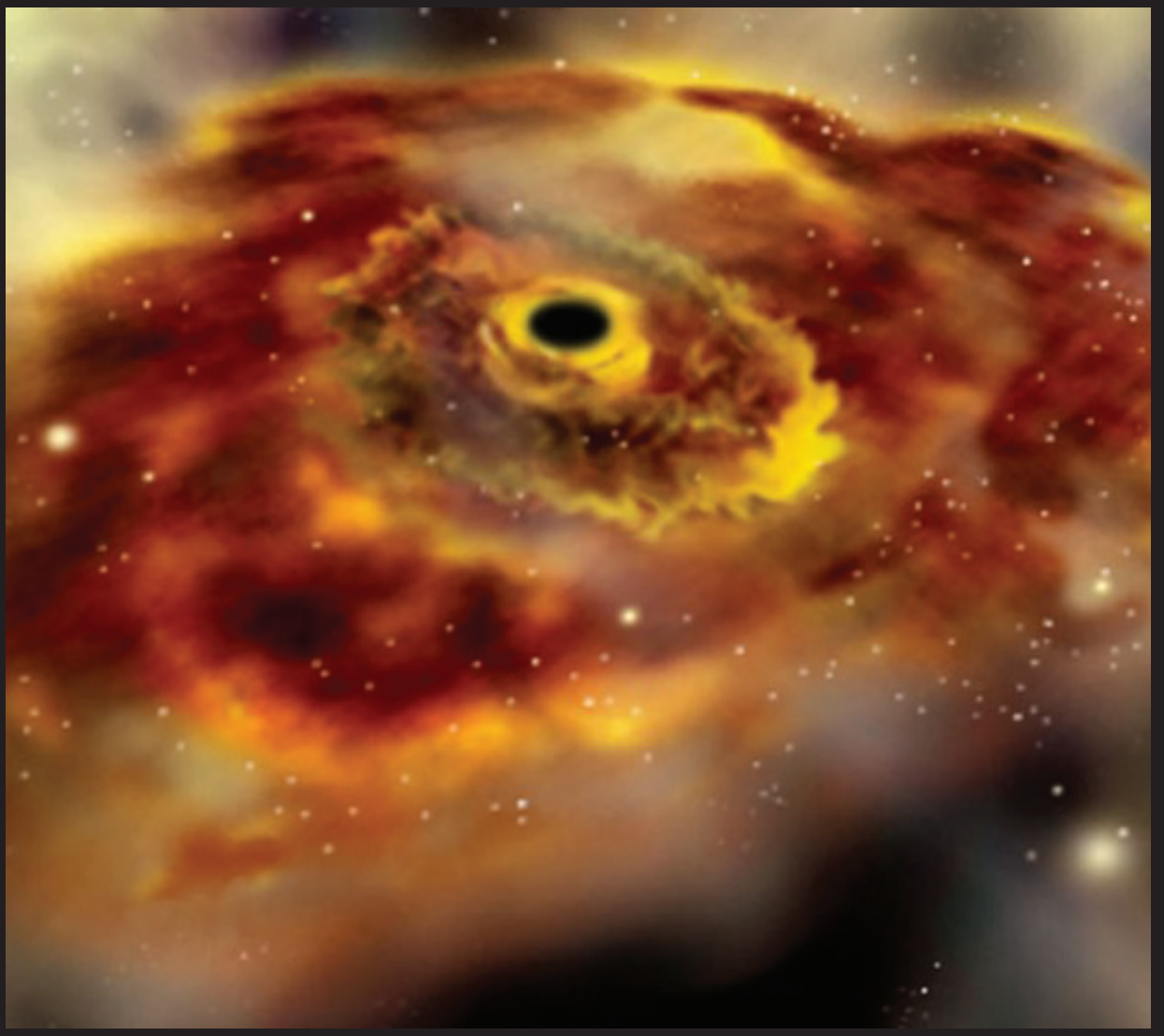
National Aeronautics and Space Administration







Black Hole Math

This collection of activities is based on a weekly series of space science problems distributed to thousands of teachers during the 2004 through 2008 school years. They were intended as supplementary problems for students looking for additional challenges in the math and physical science curriculum in grades 10 through 12. The problems are designed to be "one-pagers," consisting of a Student Page and Teacher's Answer Key. This compact form was deemed very popular by participating teachers.

The topic for this collection is **Black Holes**, which is a very popular and mysterious subject among students hearing about astronomy. Students have endless questions about these exciting and exotic objects as many of you may realize! Amazingly enough, many aspects of black holes can be understood by using simple algebra and pre-algebra mathematical skills. This booklet fills the gap by presenting black hole concepts in their simplest mathematical form.

General Approach

The activities are organized according to progressive difficulty in mathematics. Students need to be familiar with scientific notation, and it is assumed that they can perform simple algebraic computations involving exponentiation and square roots, and have some facility with calculators. The assumed level is that of Grade 10–12 Algebra II, although some problems can be worked by Algebra I students. Some of the issues of energy, force, space and time may be appropriate for students taking high school physics.

This booklet was created by the NASA Space Math program

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For more weekly classroom activities about astronomy and space science, visit
http://spacemath.gsfc.nasa.gov
Add your email address to our mailing list by contacting Dr. Sten Odenwald at
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Cover credits: ***Black hole magnetic field (XMM/Newton); Accretion disk (April Hobart, NASA/Chandra) Accretion disk (A. Simonnet, Sonoma State University, NASA Education and Public Outreach); Galactic Center x-ray (NASA/Chandra)***

Inside Credits: ***3) Black hole magnetic field XMM/Newton); 4) tidal disruption (XMM/Newton); 5) Milky Way center (NASA/Chandra) Infrared (ESA/NAOS); 6) Accretion disk (M. Weiss, NASA/Chandra); 7) accretion disk (April Hobart NASA/Chandra); 8) Accretion disk (M. Weiss NASA/Chandra); 10) accretion disk (M. Weiss NASA/Chandra); 11) x-ray emission (Ann Field, STScI);***

Teachers continue to look for ways to make math meaningful by providing students with problems and examples demonstrating its applications in everyday life. Space mathematics offers math applications through one of the strongest motivators—space. Technology makes it possible for students to *experience* the value of math, instead of just reading about it. Technology is essential to mathematics and science for such purposes as *"access to outer space and other remote locations, sample collection and treatment, measurement, data collection and storage, computation, and communication of information."* The National Council of Teachers of Mathematics (NCTM) standards include the statement that "*Similarity also can be related to such real-world contexts as photographs, models, projections of pictures,"* which can be an excellent application for black hole data.

Black Hole Math is designed to be used as a supplement for teaching mathematical topics. The problems can be used to enhance understanding of the mathematical concept or as a good assessment of student mastery.

An integrated classroom technique provides a challenge in math and science classrooms, through a more intricate method for using **Black Hole Math**. Read the scenario that follows:

Ms. Green decided to pose a Mystery Math Activity for her students. She challenged each student with math problem from the Black Hole Space Math book. She wrote the problems on the board for students to solve upon entering the classroom; she omitted the words "black hole" from each problem. Students had to solve the problem correctly in order to make a guess to solve the mystery. If the student got the correct answer, they received a free math homework pass for that night. Since the problems are a good math review prior to the end of the year final exam, all students had to do all of the problems, even if they guessed the correct answer.

Black Hole Math can be used as a classroom challenge activity, assessment tool, enrichment activity, or in a more dynamic method as is explained in the above scenario. It is completely up to the teacher, their preference, and allotted time. What it does provide, regardless of how it is used in the classroom, is the need to be proficient in math. Math proficiency is needed especially in our world of advancing technology and physical science.

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Alignment with Standards

The problems have been developed to meet specific math and science benchmarks as stated in the NSF Project 2061. Project 2061's benchmarks are statements of what all students should know or be able to do in science, mathematics, and technology by the end of grades 2, 5, 8, and 12.

The Physical Setting—The Universe (4A/H4)
Grade 12: Mathematical models and computer simulations are used in studying evidence from many sources in order to form a scientific account of the universe (see problems 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11).

Forces of Nature (4G/H1)
Grade 12: Gravitational force is an attraction between masses. The strength of the force is proportional to the masses and weakens rapidly with increasing distance between them (see problems 5, 6, 7, 8, 10, and 11).

The Mathematical World—Symbolic Relationships (9B/H3)
Grade 12: Any mathematical model, graphic or algebraic, is limited in how well it can represent how the world works. The usefulness of a mathematical model for predicting may be limited by uncertainties in measurements, by neglect of some important influences, or by requiring too much computation (see problems 2, 5, 6, and 11).

A Short Introduction to Black Holes

The basic idea of a black hole is simply an object whose gravity is so strong that light cannot escape from it. It is black because it does not reflect light, nor does its surface emit any light.

Before Princeton Physicist John Wheeler coined the term black hole in the mid-1960s, no one outside of the theoretical physics community really paid this idea much attention.

In 1798, the French mathematician Pierre Laplace first imagined such a body using Newton's Laws of Physics (the three laws plus the Law of Universal Gravitation). His idea was very simple and intuitive. We know that rockets have to reach an escape velocity in order to break free of Earth's gravity. For Earth, this velocity is 11.2 km/sec (40,320 km/hr or 25,000 miles/hr). Now let's add enough mass to Earth so that its escape velocity climbs to 25 km/sec…2000 km/sec…200,000 km/sec, and finally the speed of light: 300,000 km/sec. Because no material particle can travel faster than light, once a body is so massive and small that its escape velocity equals light-speed, it becomes dark. This is what Laplace had in mind when he thought about “black stars.” This idea was one of those idle speculations at the boundary of mathematics and science at the time, and nothing more was done with the idea for over 100 years.

Once Albert Einstein had completed developing his Theory of General Relativity in 1915, the behavior of matter and light in the presence of intense gravitational fields was revisited. This time, Newton's basic ideas had to be extended to include situations in which time and space could be greatly distorted. There was an intense effort by mathematicians and physicists to investigate all of the logical consequences of Einstein's new theory of gravity and space. It took less than a year before one of the simplest kinds of bodies was thoroughly investigated through complex mathematical calculations.

The German mathematician Karl Schwarzschild investigated what would happen if all the matter in a body were concentrated at a mathematical point. In Newtonian physics, we call this the center of mass of the body. Schwarzschild chose a particularly simple body: one that was a perfect sphere and not rotating at all. Mathematicians such as Roy Kerr, Hans Reissner, and Gunnar Nordstrom would later work out the mathematical details for other kinds of black holes.

Schwarzschild black holes are actually very simple. Mathematicians even call them elegant because their mathematics is so compact, exact, and beautiful. They have a geometric feature called an “event horizon” (Problem 1) that mathematically distinguishes the inside of the black hole from the outside. These two regions have very different geometric properties for the way that space and time behave. The world outside the event horizon is where we live and contains our universe, but inside the event horizon, space and time behave in very different ways entirely (Problem 9). Once inside, matter and light cannot get back out into the rest of the universe. This horizon has nothing to do, however, with the Newtonian idea of an escape velocity.

By the way, these statements sound very qualitative and vague to students, but the mathematics that goes into making these statements is both complex and exact. With this in mind, there are four basic kinds of black hole solutions to Einstein's equations:

Schwarzschild: These are spherical and do not rotate. They are defined only by their total mass.

Reissner-Nordstrom: These possess mass and charge but do not rotate.

Kerr: These rotate and are flattened at the poles, and only described by their mass and amount of spin (angular momentum).

Kerr-Nordstrom: These possess mass and charge, and they rotate.

There are also other types of black holes that come up when quantum mechanics is applied to understanding gravity or when cosmologists explore the early history of the universe. Among these are

Planck-Mass: These have a mass of 0.00000001 kilograms and a size that is 100 billion billion times smaller than a proton.

Primordial: These can have a mass greater than 10 trillion kilograms and were formed soon after the big bang and can still exist today. Smaller black holes have long-since vanished through evaporation in the time since the big bang.

A Common Misconception

Black holes cannot suck matter into them except under certain conditions. If the sun turned into a black hole, Earth and even Mercury would continue to orbit the new sun and not fall in. There are two common cases in the universe in which matter can be dragged into a black hole. Case 1: If a body orbits close to the event horizon in an elliptical orbit, it emits gravitational radiation, and its orbit will eventually decay in millions of years. Case 2: A disk of gas can form around a black hole, and through friction, matter will slowly slide into the black hole over time.

How Black Holes are Formed

Black holes can come in any size, from microscopic to supermassive. In today's universe, massive stars detonate as supernovae and this can create stellar-mass black holes (1 solar mass = 1.9×10^{30} kg). When enough of these are present in a small volume of space, like the core of a globular cluster, black holes can absorb each other and in principle, can grow to several hundred times the mass of the sun. If there is enough matter (i.e., gas, dust, and stars) for a black hole to “eat,” it can grow even larger. There is a black hole in the star-rich core of the Milky Way that has a mass equal to nearly 3 million suns. The cores of more massive and distant galaxies have supermassive black holes containing the equivalent of 100 million to as much as 10 billion suns. Astronomers are not entirely sure how these supermassive black holes evolved so quickly to their present masses given that the universe is only 14 billion years old.

Currently, there are no known ways to create black holes with masses less than about 0.1 times the sun's mass, and through a speculative process called Hawking Radiation, black holes less than 1 trillion g in mass would have evaporated by now if they had formed during the Big Bang.

A Short List of Known Black Holes

Stellar-Mass

Name	Constellation	Distance (light years)	Mass (in solar units)
Cygnus X–1	Cygnus	7000	16
SS 433	Aquila	16,000	11
Nova Mon 1975	Monocerous	2700	11
Nova Persi 1992	Perseus	6500	5
IL Lupi	Lupus	13,000	9
Nova Oph 1977	Ophiuchus	33,000	7
V4641 Sgr	Sagittarius	32,000	7
Nova Vul 1988	Vulpecula	6500	8
V404 Cygni	Cygnus	8000	12

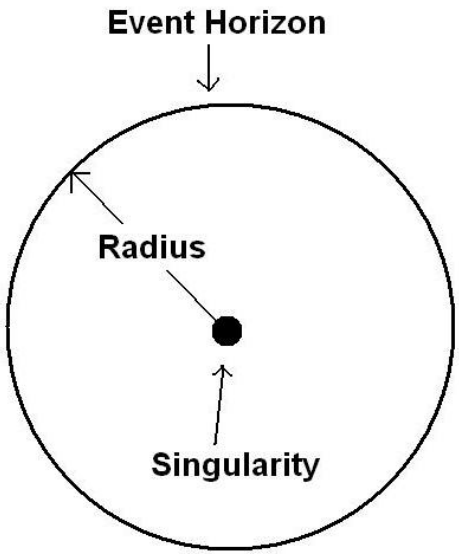
Note: The mass is the sum of the companion star and the black hole masses. 16 means 16 times the mass of the sun.

Galactic-Mass

Name	Constellation	Distance (light years)	Mass (in solar units)
NGC–205	Andromeda	2,300,000	90,000
Messier–33	Triangulum	2,600,000	50,000
Milky Way SgrA*	Sagittarius	27,000	3,000,000
Messier–31	Andromeda	2,300,000	45,000,000
NGC–1023	Canes Venatici	37,000,000	44,000,000
Messier–81	Ursa Major	13,000,000	68,000,000
NGC–3608	Leo	75,000,000	190,000,000
NGC–4261	Virgo	100,000,000	520,000,000
Messier–87	Virgo	52,000,000	3,000,000,000

Note: The first three are called Intermediate-mass black holes. The remaining are called supermassive.

The Event Horizon



Black holes are objects that have such intense gravitational fields, they do not allow light to escape from them. They also make it impossible for anything that falls into them to escape, because to do so, they would have to travel at speeds faster than light. No forms of matter or energy can travel faster than the speed of light, so that is why black holes are so unusual!

There are three parts to a simple black hole:

Event Horizon - Also called the Schwarzschild radius, that's the part that we see from the outside. It looks like a black, spherical surface with a very sharp edge in space.

Interior Space - This is a complicated region where space and time can get horribly mangled, compressed, stretched, and otherwise a very bad place to travel through.

Singularity - That's the place that matter goes when it falls through the event horizon. It's located at the center of the black hole, and it has an enormous density. You will be crushed into quarks long before you get there!

Black holes can, in theory, come in any imaginable size. The size of a black hole depends on the amount of mass it contains. It's a very simple formula, especially if the black hole is not rotating. These 'non-rotating' black holes are called Schwarzschild Black Holes.

Equation 1) $$R = \frac{2GM}{c^2}$$

Equation 2) $$R = 1.48x10^{-27} M$$

Problem 1 - The two formulas above give the Schwarzschild radius, R, of a black hole in terms of its mass, M. From Equation 1, verify Equation 2, which gives R in meters and M in kilograms, using c= 3 x 10^8 m/sec for the speed of light, and G = 6.67x10^{-11} Newtons m^2/kg^2 for the gravitational constant.

Problem 2 - Calculate the Schwarzschild radius, in meters, for Earth where
M = 5.7 x 10^{24} kilograms.

Problem 3 - Calculate the Schwarzschild radius, in kilometers, for the sun, where
M = 1.9 x 10^{30} kilograms.

Problem 4 - Calculate the Schwarzschild radius, in kilometers, for the entire Milky Way, with a mass of 250 billion suns.

Problem 5 - Calculate the Schwarzschild radius, in meters, for a black hole with the mass of an average human being with M = 60 kilograms.

$$T = t\sqrt{1 - \frac{2GM}{Rc^2}}$$

T = the time measured by someone located on a planet (seconds)

t = the time measured by someone located in space (seconds)

M = the mass of the planet (kg)

R = the distance to the far-away observer from the planet (m)

And the natural constants are:
$G = 6.67 \times 10^{-11}$ Nt m^2/kg^2
$C = 3 \times 10^8$ m/sec

The modern theory of gravity, called the Theory of General Relativity, developed by Albert Einstein in 1915 leads to some very unusual predictions, which have all been verified by experiments.

One of the strangest ones is that two people will experience the passage of time very differently if one is standing on the surface of a planet, and the other one is in space. This is because the rate of time passing depends on the strength of the gravitational field that the observer is in.

For example, at the surface of a very dense neutron star, R = 20 km and $M = 1.9 \times 10^{30}$ kg, so

$$T = t\ (1 - 0.15)^{1/2} = 0.92\ t$$

This means that for every hour that goes by on the surface of the neutron star (T = 60 minutes), someone in space will see t = 60 / 0.92 = 65 minutes pass from a vantage point in space.

The following problems require accuracy to the 11th decimal place. Most hand calculators only provide 9 digits. Students may use the 'calculator' accessory provided on all PCs and Macs.

Problem 1 - The GPS satellites orbit Earth at a distance of R = 26,560 km. If the mass of Earth is 5.9×10^{24} kg, use the formula to determine the time dilation factor.

Problem 2 - What is the time dilation factor at Earth's surface?

Problem 3 - What is the ratio of the dilation in space to the dilation at earth's surface?

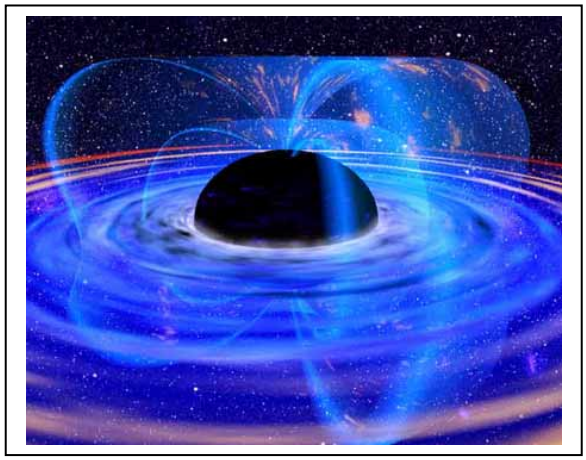
Problem 4 - At the speed of light (3×10^8 m/sec) how long does it take a radio signal from the GPS satellite to travel 26,560 km to a hand-held GPS receiver?

Problem 5 - The excess time delay between a receiver at Earth's surface, and the GPS satellite is defined by the ratio computed in Problem 3, multiplied by the total travel time in Problem 4. What is the time delay for the GPS-Earth system?

Problem 6 - From your answer to Problem 5, how much extra time does the radio signal take compared to your answer to Problem 4?

Problem 7 - At the speed of light, how far will the radio signal travel during the extra amount of time?

Time Dilation Near a Black Hole



Artists illustration of a black hole with an orbiting disk of gas and dust. Friction in the disk causes matter to steadily flow inwards until it reaches the black hole event horizon. Magnetic forces in the disk cause matter to flow in complex jets and plumes. Time dilation causes delays in events taking place near the black hole compared to what distant observers will record.

Time dilation near a black hole is a lot more extreme than what the GPS satellite network experiences in orbit around Earth.

$$T = t\sqrt{1 - \frac{2GM}{Rc^2}}$$

T = time measured by someone located on a planet (sec)

t = time measured by someone located in space (sec)

M = mass of the planet (kg)

R = distance to the far-away observer from the planet (m)

Problem 1 - In the time dilation formula above, evaluate the quantity $2GM/c^2$ for a black hole with a mass of one solar mass (1.9×10^{30} kg), and convert the answer to kilometers to two significant figures.

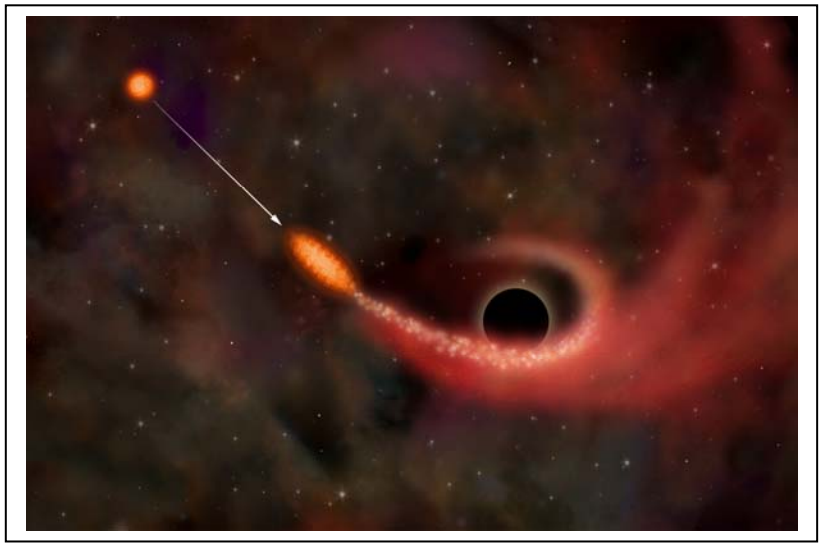
Problem 2 - Re-write the formula in a more tidy form using your answer to Problem 1.

Problem 3 - In the far future, a scientific outpost has been placed in orbit around this solar-mass black hole at a distance of 10 km. What will the time dilation factor be at this location?

Problem 4 - A series of clock ticks were sent out by the satellite once each hour. What will be the time interval in seconds between the clock ticks by the time they reach a distant observer?

Problem 5 - If one tick arrived at 1:00 PM at the distant observer, when will the next clock tick arrive?

Problem 6 - A radio signal was sent by the black hole outpost to a distant observer. At the frequency of the signal, when transmitted from the outpost, the individual wavelengths take 0.000001 seconds to complete one cycle. From your answer to Problem 3, how much longer will they take by the time they arrive at the distant observer?



Thanks to two orbiting X-ray observatories, astronomers have the first strong evidence of a supermassive black hole ripping apart a star and consuming a portion of it. The event, captured by NASA's Chandra and ESA's XMM-Newton X-ray Observatories, had long been predicted by theory, but never confirmed until now. Giant black holes in just the right mass range would pull on the front of a closely passing star much more strongly than on the back. Such a strong tidal force would stretch out a star and likely cause some of the star's gasses to fall into the black hole. The infalling gas has been predicted to emit just the same blast of X-rays that have recently been seen in the center of galaxy RX J1242-11 located 700 million light years from the Milky Way, in the constellation Virgo. (NASA news report at http://chandra.harvard.edu/photo/2004/rxj1242/

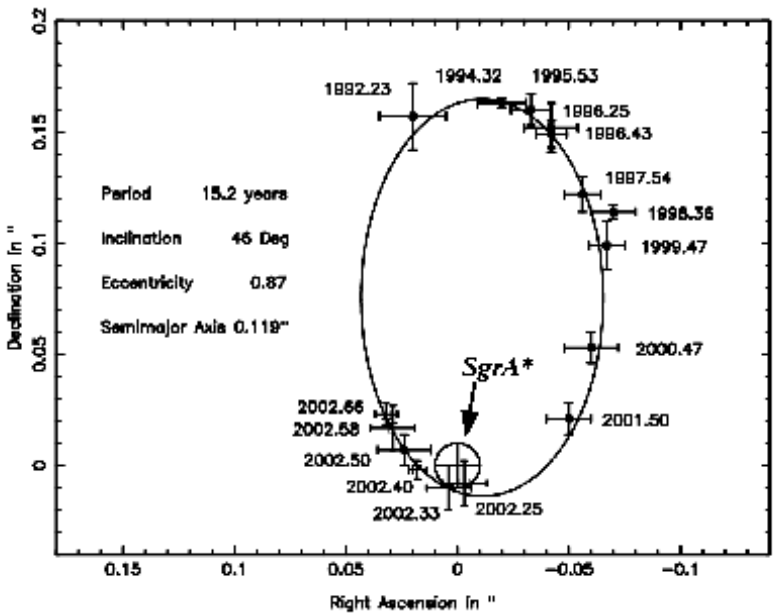
Problem 1 - The size of the event horizon of a black hole (called the Schwarschild radius) is given by the formula R = 2.8 M, where R is the radius in km, and M is that mass of the black hole in units of the sun's mass. A supermassive black hole can have a mass of 100 million times the sun. What is its Schwarschild radius in: A) kilometers, B) multiples of the Earth orbit radius called an Astronomical Unit (1 AU = 149 million km), C) compared to the orbit of Mars (1.5 AU)

Problem 2 - Black holes are one of the most efficient phenomena in converting matter into energy. As matter falls inward in an orbiting disk of gas, friction heats the gas up, and the energy released can be as much as 7% of the rest mass energy of the infalling matter. The quasar 3C273 has a power output of 3.8×10^{38} Joules/second. If $E = mc^2$ is the formula that converts mass (in kg) into energy (in Joules) and c = the speed of light, 3×10^8 m/sec, how many grams per year does this quasar luminosity imply if 1 year = 3.1×10^7 seconds?

Problem 3 - If the mass of the Sun is 1.9×10^{30} kg, how many suns per year have to be consumed by the 3C273 supermassive black hole at the black hole conversion efficiency of 7%? (Note: 7% efficiency means that for every 100 kg involved, 7 kg are converted into pure energy by $E=mc^2$)

At the center of our Milky Way Galaxy lies a black hole, called Sagittarius A*, with over 2.6 million times the mass of the Sun. Once a controversial claim, this astounding conclusion is now virtually inescapable and based on observations of stars orbiting very near the galactic center.

The Chandra image to the left shows the x-ray light from a region of space a few light years across. The black hole is invisible, but it is near the center of this image. The gas near the center produces x-ray light as it is heated. Many of the 'stars' in the field probably have much smaller black holes near them that are producing the x-ray light from the gas they are consuming.



Astronomers patiently followed the orbit of a particular star, designated S2. Their results convincingly show that S2 is moving under the influence of the enormous gravity of an unseen object, which must be extremely compact and contain huge amounts of matter yet emits no light -- a supermassive black hole. The drawing above shows the orbit shape.

Problem 1 - Kepler's Third Law can be used to determine the mass of a body by measuring the orbital period, T, and orbit radius, R, of a satellite. If R is given in units of the Astronomical Unit (AU) and T is in years, the relationship becomes $R^3 / T^2 = M$, where M is the mass of the body in multiples of the sun's mass. In these units, for Earth, R = 1.0 AU, and T = 1 year, so M = 1.0 solar masses. In 2006, the Hubble Space Telescope, found that the star Polaris has a companion, Polaris Ab, whose distance from Polaris is 18.5 AU and has a period of 30 years. What is the mass of Polaris?

Problem 2 - The star S2 orbits the supermassive black hole Sagittarius A*. Its period is 15.2 years, and its orbit distance is about 840 AU. What is the estimated mass of the black hole at the center of the Milky Way?

$$a = \frac{2\,G\,M\,d}{R^3}$$

A tidal force is a difference in the strength of gravity between two points. The gravitational field of the Moon produces a tidal force across the diameter of Earth, which causes Earth to deform. It also raises tides of several meters in the solid Earth, and larger tides in the liquid oceans.

If the tidal force is stronger than a body's cohesiveness, the body will be disrupted. The minimum distance that a satellite comes to a planet before it is shattered this way is called its Roche Distance. The artistic image to the left shows what tidal disruption could be like for an unlucky moon.

A human falling into a black hole will also experience tidal forces. In most cases these will be lethal! The difference in acceleration between the head and feet could be many thousands of Earth gravities. A person would literally be pulled apart, and his atoms drawn into a narrow string of matter! Some physicists have termed this process spaghettification!

Problem 1 - The equation lets us calculate the tidal acceleration, **a**, across a body with a length of **d**. The acceleration of gravity on Earth's surface is 9.8 m/sec^{2}. The tidal acceleration between your head and feet is given by the above formula. For M = the mass of Earth (5.9 x 10^{24} kg), R = the radius of Earth (6.4 x 10^{6} m) and the constant of gravity whose value is G = 6.67 x 10^{-11} Nt m^{2}/kg^{2} calculate the tidal acceleration, a, if d = 2 meters.

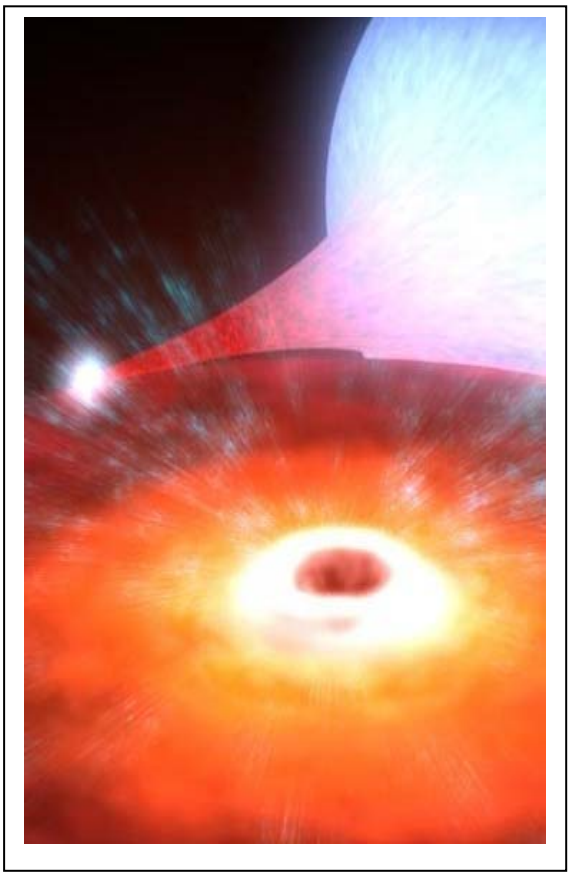
Problem 2 - What is the tidal acceleration across the full diameter of Earth?

Problem 3 - A stellar-mass black hole has the mass of the sun (1.9 x 10^{30} kg), and a radius of 2.8 km. What would be the tidal acceleration across a human at a distance of 100 km?

Problem 4 - A supermassive black hole has 100 million times the mass of the sun (1.9 x 10^{38} kg), and a radius of 280 million km. What would be the tidal acceleration near the event horizon of the supermassive black hole?

Problem 5 - Which black hole could a human enter without being spaghettified?

Falling Into a Black Hole



An object that falls into a black hole will cross the event horizon, and speed up as it gets closer. This is like a ball traveling faster and faster as it is dropped from a tall building. Suppose the particle fell from infinity. How fast would it be traveling? We can answer this question by considering the concepts of kinetic energy (K.E.) and gravitational potential energy (P.E.):

$$K.E = \frac{1}{2}mV^2 \qquad \text{and} \qquad P.E. = \frac{GMm}{R}$$

The kinetic energy that the particle with mass, m, will gain as it falls, will depend on the total potential energy it has lost in traveling from infinity to a distance R. By setting the two equations equal to each other, we can relate the kinetic energy a particle gains as it falls to its current distance of R from the center of mass. The quantity, M, is the mass of the gravitating body the particle is falling towards. G is the constant of gravitation which equals 6.67 x 10^{-11} Nt m^2/kg^2

$$\frac{1}{2}mV^2 = \frac{GMm}{R}$$

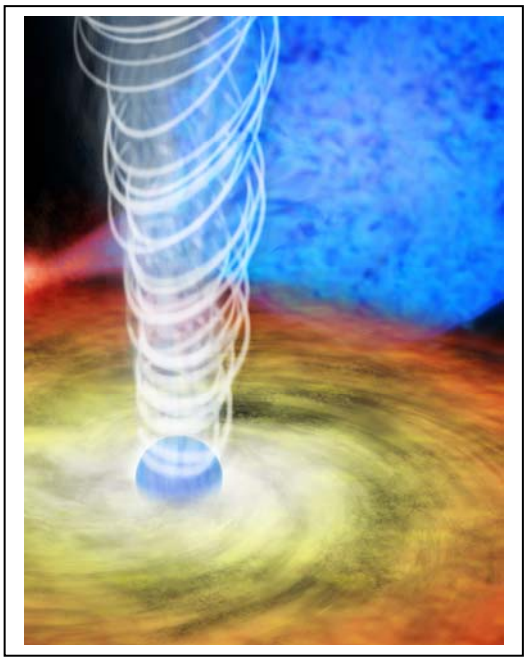
We can then solve for the speed, V, in terms of R

$$V = \sqrt{\frac{2GM}{R}}$$

Problem 1 - Suppose a body falls to Earth and strikes the ground. How fast will it be traveling when it hits if M = 5.9 x 10^{24} kg and R = 6,378 km? Explain why this is the same as Earth's escape velocity?

Problem 2 - NASA's ROSSI satellite was used in 2004 to determine the mass and radius of a neutron star in the binary star system named EXO 0748-676, located about 30,000 light-years away in the southern sky constellation Volans, or the Flying Fish. The neutron star was deduced to have a mass of 1.8 times the sun, and a radius of 11.5 km. A) How fast, in km/sec, will a particle strike the surface of the neutron star if the mass of the sun is 1.9 x 10^{30} kg? B) In terms of percentage, what will be the speed compared to the speed of light: 300,000 km/sec?

Problem 3 - The star HD226868 is a binary star with an unseen companion. It is also the most powerful source of X-rays in the sky second to the sun - it's called Cygnus X-1. Astronomers have determined the mass of this companion to be 8.7 times the sun. As a black hole, its event horizon radius would be R = 2.8 x 8.7 = 24 km. A) How fast, in km/sec, would a body be traveling as it passed through the event horizon? B) In percentage compared to the speed of light?



An artist's impression of a black hole orbiting a companion star, and gravitationally attracting gas from the star into an orbiting accretion disk. Through friction, the gas becomes hotter as it approaches the black hole, turning from red to yellow to white. Most of the gas is swallowed by the black hole, but some is magnetically launched in jets away from the black hole at almost the speed of light. (Credit: M. Weiss, NASA/Chandra)

The farther a particle falls towards a black hole, the faster it travels, and the more kinetic energy it has. Kinetic energy is mathematically defined as K.E. = 1/2 m V^2 where m is the mass of the particle and V is its speed.

Suppose all this energy is converted into heat energy by friction as the particle falls, and that this added energy causes nearby gases to heat up. How hot will the gas get? The equivalent amount of thermal energy, T.E., carried by a single particle is

$$T.E. = \frac{3}{2}kT$$

where Boltzman's Constant k = 1.38 x 10^{-23} Joules/deg. If we set K.E = T.E we get

$$T = \frac{mV^2}{3k}$$

If all the particles in a gas carried this same kinetic energy, then we would say the gas has a temperature of T degrees Kelvin. We also know that the potential energy of the particle is given by

$$P.E. = \frac{GMm}{R}$$

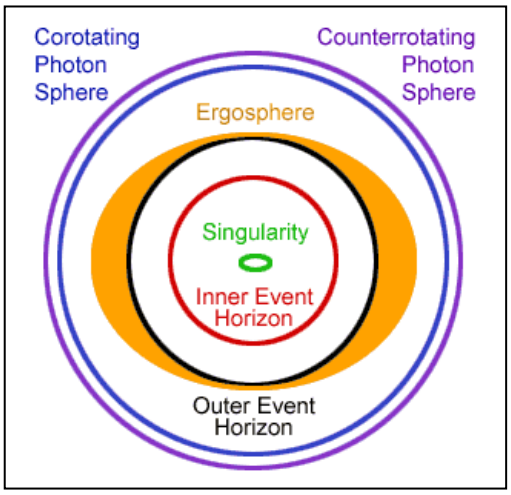
So if we set P.E = T.E we also get the temperature

$$T = \frac{2GMm}{3kR}$$

Problem 1 - The formula T = 2 G M m/(3kR) gives the approximate temperature of hydrogen gas (m = 1.6 x 10^{-27} kg) in an accretion disk around a black hole. To two significant figures, what is the temperature for the material at the distance of Earth's orbit for a solar-mass black hole? (R = 1.47 x 10^{11} m, M = 1.9 x 10^{30} kg, for the constant of gravity G = 6.67 x 10^{-11} Nt m^2/kg^2)?

Problem 2 - How hot would the disk be at the distance of Neptune (R = 4.4 x 10^{12} meters)?

Problem 3 - X-rays are the most common forms of energy produced at temperatures above 100,000 K. Visible light is produced at temperatures above 2,000 K. Infrared radiation is commonly produced for temperatures below 500 K. What would you expect to see if you studied the accretion disk around a solar-mass sized black hole?



Outside a black hole, we have the normal universe of space, time, matter and energy we have all come to know. But inside, things are very different. We know this because the same mathematics that predicts black holes should exist, also predicts what to find inside them.

One of the biggest surprises is the way that time and space, themselves, behave.

The figure to the left shows a few of the regions that have been identified from the mathematics of rotating black holes.

Problem 1 - In relativity, space and time are part of a single 4-dimensional thing called spacetime. There are 3 dimensions to space and 1-dimension to time. Every point, called an event, has three coordinates to describe its location in space, and one extra coordinate to describe its location in time. We write these as an ordered set like A(x,y,z,t) or B(x,y,z,t).

Write the ordered set for the following event called A: I travel east 3 miles, north 5 miles, and up 1 miles, at 9:00 a.m on February 16, 2008. (Use x along East-West, y along North-South, z along Up-Down, with East, North and Up directions positive.)

Problem 2 - Suppose you travel from one event with coordinates A(3 km, 6 km, 2 km, 5:00 p.m) to another event B(5 km, 7 km, 3 km, 8:00 p.m). How far did you travel in space in each direction during the time interval from 5:00 p.m to 8:00 p.m?

Problem 3 - Use the Pythagorean Theorem to calculate the actual distance in space in Problem 2.

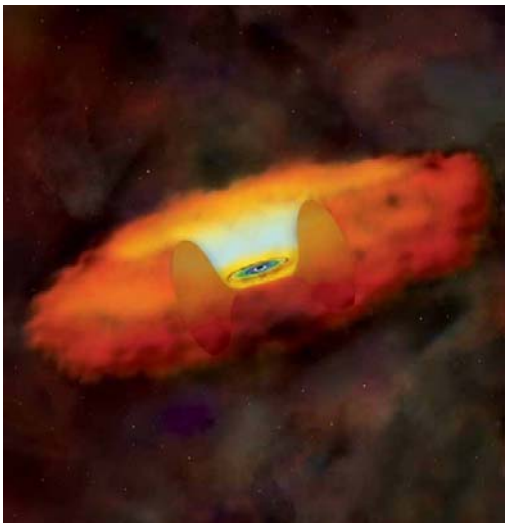
Problem 4 - The 4-dimensional, hyperdistance between the events is found by using the hyperbolic Pythagorean Theorem formula $D^2 = -c^2T^2 + x^2 + y^2 + z^2$ where c is the speed of light (c = 300,000 km/sec). Calculate the hyperdistance, D^2, between Events A and B in Problem 2.

Problem 5 - Based on your answer to Problem 4, which part of D^2 makes the largest contribution to the hyperdistance, the time-like part, T, or the space-like part (x,y,z)?

Problem 6 - Inside a black hole, the formula for D^2 changes to $D^2 = c^2T^2 - x^2 - y^2 - z^2$ Suppose Events A and B are now happening inside the black hole. What is the hyper-distance between them, and does the space or time-like part make the biggest contribution?

Problem 7 - If an observer defines 'time' as the part of D^2 that has a negative sign, and 'space' as the part that has the positive sign, can you explain what happens as the traveler passes inside the black hole?

Black Hole Power



Black holes are sometimes surrounded by a disk of orbiting matter. This disk is very hot. As matter finally falls into the black hole from the inner edge of that disk, it releases about 7% of its rest-mass energy in the form of light. Some of this energy was already lost as the matter passed through, and heated up, the gases in the surrounding disk. But the over-all energy from the infalling matter is about 7% of its rest-mass in all forms (heat+ light).

The power produced by a black hole is phenomenal, with far more energy per kilogram being created than by ordinary nuclear fusion, which powers the sun.

A black hole accretion disk (M. Weiss NASA/Chandra)

Problem 1 - The event horizon of a black hole has a radius of R = 2.8 M kilometers, where M is the mass of the black hole in multiples of the sun's mass. Assume the event horizon is a spherical surface, so its surface area is $S = 4 \pi R^2$. What is the surface area of A) a stellar black hole with a mass of 10 solar masses? B) a supermassive black hole with a mass of 100 million suns?

Problem 2 - What is the volume of a spherical shell with the surface area of the black holes in Problem 1, with a thickness of one meter?

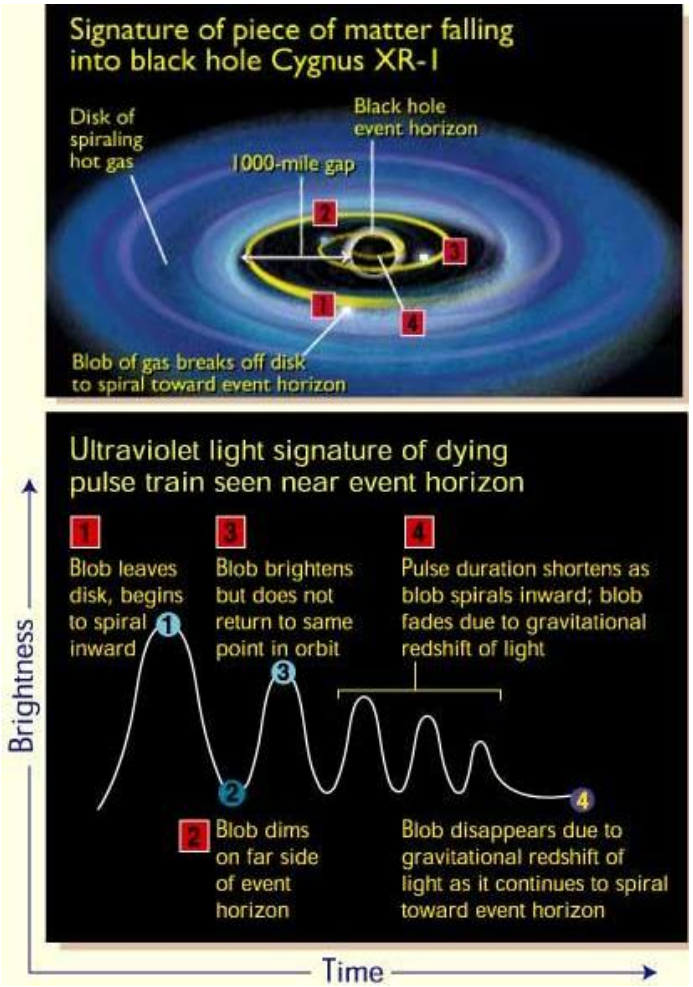
Problem 3 - If the density of gas near the horizon is 10^{16} atoms/meter^3 of hydrogen, how much matter is in each black hole shell, if the mass of a hydrogen atom is 1.6×10^{-27} kg?

Problem 4 - If $E = m c^2$ is the rest mass energy, E, in Joules, for a particle with a mass of m in kg, what is the rest mass energy equal to the masses in Problem 3 if $c = 3 \times 10^8$ m/sec is the speed of light and only 7% of the mass produced energy?

Problem 5 - Suppose the material was traveling at 1/2 the speed of light as it crossed the event horizon, how much time does it take to travel one meter if $c = 3 \times 10^8$ m/sec is the speed of light?

Problem 6 - The power produced is equal to the energy in Problem 4, divided by the time in Problem 5. What is the percentage of power produced by each black hole compared to the sun's power of 3.8×10^{26} Joules/sec?

Black Hole…Fade-out!



As seen from a distance, not only does the passage of time slow down for someone falling into a black hole, but the image fades to black!

This happens because, during the time that the object reaches the event horizon and passes beyond, a finite number of light particles (photons) will be emitted. Once these have been detected to make an image, there are no more left because the object is on the other side of the event horizon and the photons cannot escape. A star, collapsing to a black hole, will be going very fast as it collapses, then appear to slow down as time dilates. Meanwhile, the image will become redder and redder, until it literally fades to black!

Photographs taken by the Hubble Space Telescope of the black-hole candidate called Cygnus XR-1 detected two instances where a hot gas blob appeared to be slipping past the event horizon for the black hole. Because of the gravitational stretching of light, the fragment disappeared from Hubble's view before it ever actually reached the event horizon. The pulsation of the blob, an effect caused by the black hole's intense gravity, also shortened as it fell closer to the event horizon. Without an event horizon, the blob of gas would have brightened as it crashed onto the surface of the accreting body. See The Astrophysical Journal, 502:L149–L152, 1998 August 1. (Diagram courtesy Ann Field: STScI)

$$L = L_0 e^{\frac{-2T}{3\sqrt{3}\ 2M}}$$

Problem 1 - The exponential formula above predicts the decay of the light from matter falling in to a black hole. T is the time in seconds measured by distant observer, and M is the mass of the black hole in units of solar masses. How long does it take for the light to fall to half its initial luminosity (i.e. power in units of watts) given by L_0 for a M =1.0 solar mass, stellar black hole?

Problem 2 - How long will your answer be, in years, for a supermassive black hole with M = 100 million times the mass of the Sun?

Problem 3 - The supermassive black hole in Problem 2 'swallows' a star. If the initial luminosity, L_0, of the star is 2.5 times the Sun's, to two significant figures,how long will it take before the brightness of the star fades to 0.0025 Suns, and can no longer be detected from Earth?

Additional Mathematical Resources About Black Holes

It is a challenge to find mathematical resources about black holes that are not too advanced, but that still give the student and teacher some idea of how to think about them more quantitatively. Ironically, if you GOOGLE 'black hole math' you will quickly discover that this very math guide is among the Top-7 options! Clearly there is a big need for this kind of resource that can be used by the K-12 community. Here are just a few others that might be helpful:

Death Spiral Around a Black Hole - Hubble Discovery
http://hubblesite.org/newscenter/archive/releases/2001/03

Chandra Observatory Detects Event Horizon -
http://chandra.harvard.edu/photo/2001/blackholes/

Ask the Astronomer: 87 FAQs About Black Holes:
http://www.astronomycafe.net/qadir/abholes.html

Imagine the Universe: Black Hole FAQs
http://imagine.gsfc.nasa.gov/docs/ask_astro/black_holes.html

New Evidence for Black Holes from NASA
http://science.nasa.gov/headlines/y2001/ast12jan_1.htm

A trip into a black hole
http://antwrp.gsfc.nasa.gov/htmltest/rjn_bht.html

A note from the Author,

Since they first came into public view in the early 1970s, black holes have been a constant source of curiosity and mystery for millions of adults and children. No astronomer has had the experience of visiting a classroom, and NOT being asked questions about these weird objects with which we share our universe.

Beyond answering that they are objects with such intense gravity that even light cannot escape them, we tend to be at a loss for what to say next. The mathematics of Einstein's General Theory of Relativity are extremely complex even for advanced undergraduates in mathematics, so we tend to resort to colorful phrases and actual hand-waving to describe them to eager students.

Actually, there are many important aspects of black holes that can be readily understood by using pre-algebra (scientific notation), Geometry (concepts of space and coordinates, Pythagorean Theorem), Algebra I (working with simple formulae), and Algebra II (working with asymptotic behavior).

This book is a compilation of some of my favorite elementary problems in black hole physics. They will introduce the student to the important concept of the event horizon, time dilation, and how energy is extracted from a black hole to create many kinds of astronomical phenomena. Some of these problems may even inspire a student to tackle a Science Fair or Math Fair problem!

Black holes are indeed something of a mystery, but many of their most well-kept secrets can be understood with just a little mathematics. I hope the problems in this book will inspire students to learn more about them!

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