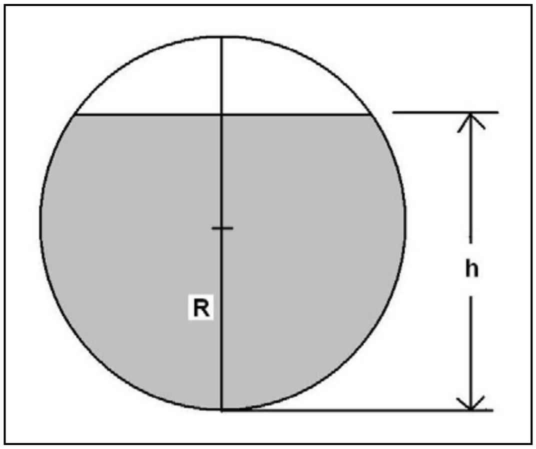
Fuel Level in a Spherical Tank



Spherical tanks are found in many different situations, from the storage of cryogenic liquids, to fuel tanks. Under the influence of gravity, or acceleration, the liquid will settle in a way such that it fills the interior of the tank up to a height, h. We would like to know how full the tank is by measuring h and relating it to the remaining volume of the liquid. A sensor can then be designed to measure where the surface of the liquid is, and from this derive h.

Problem 1 - Slice the fluid into a series of vertically stacked disks with a radius r(h) and a thickness dh. What is the general formula for the radius of each disk?

Problem 2 - Set up the integral for the volume of the fluid and solve the integral.

Problem 3 - Assume that fluid is being withdrawn from the tank at a fixed rate $dV/dt = -F$. What is the equation for the change in the height of the fluid volume with respect to time? A) Solve for the limits $h \ll R$ and $h \gg R$. B) Solve graphically for R=1 meter, F=100 cm^3/min. (Hint: select values for h and solve for t).

Answer Key

Problem 1 - $r(h)^2 = R^2 - (R-h)^2$ so $r(h)^2 =$ $\mathbf{2Rh - h^2}$

Problem 2 - The integrand will be $\pi (2Rh - h^2)\, dh$ and the solution is $\mathbf{\pi Rh^2 - 1/3\pi h^3}$

Problem 3 -

$$\frac{dV}{dt} = 2\pi R h \frac{dh}{dt} - \pi h^2 \frac{dh}{dt} \quad \text{so} \quad \frac{dV}{dt} = (2\pi R h - \pi h^2)\frac{dh}{dt} = -F$$

Then

$$\frac{dh}{dt} = \frac{-F}{(2\pi R h - \pi h^2)}$$

The integrands become: $(2\pi R h - \pi h^2)\, dh = -F\, dt$. This can be integrated from t=0 to t = T to obtain $\pi R h^2 - 1/3\, \pi h^3 = -F T$ and simplified to get

$$\mathbf{h^3 - 3Rh^2 - (3FT)/\pi = 0}$$

We would normally like to invert this equation to get h(T), but cubic equations of the form $x^3 - \alpha x^2 + \beta = 0$ cannot be solved analytically. We can solve it for two limiting cases. Case 1 for a tank nearly empty where h << R. This yields $\mathbf{h(T) = (F T/R)^{1/2}}$ Case 2 is for a tank nearly full so that h >> R, and we get $h^3 = 3 F T/\pi$ and $\mathbf{h(T) = (3 F T/\pi)^{1/3}}$. The full solution for h(T) can be solved graphically. Since R is a constant, we can select a new variable U = h/R and re-write the equation in terms of the magnitude of h relative to the radius of the tank.

$U^3 - 3U^2 = (3 F T)/\pi R^3$ and plot this for selected combinations of (U,T) where time, T, is the dependent variable. The solution below is for F = 100 cm^3/minute, R = 1 meter, with the intervals in h spaced 10 cm. The plot was generated using an *Excel* spreadsheet.